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VIBRATION EFFECTS ON HEAT TRANSFER IN CRYOGENIC SYSTEMS

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MISSISSIPPI STATE UNIVERSITY
STATE COLLEGE, MISSISSIPPI

Department of Mechanical Engineering
Mississippi State University
State College, Mississippi 39762

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By

E. H. Bishop, Associate Professor
C. J. Bell, Associate Professor
C. T. Carley, Associate Professor
T. T. Crow, Professor
R. E. Forbes, Graduate Assistant
E. W. Wellborn, Graduate Assistant
E. W. Farrar, Graduate Assistant

ET AL

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Approved By:

G. S. Holmes

A. G. Holmes, Head
Mechanical Engr. Department

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NOMENCLATURE

A	$(=H'/W')$ Aspect ratio
a	Acceleration in g's
a'	Maximum amplitude of vibration
B'_y	Body force
C	Relative compressibility
c'	Specific heat
D'	Enclosure depth
E	Internal energy
f	Frequency of vibration
g	Acceleration of gravity
g'	$(=a'\omega'^2)$ Maximum enclosure acceleration
g'_o	Acceleration of gravity
H'	Enclosure height
h'	Convective heat transfer coefficient
k	Boltzmann's constant
k'	Thermal conductivity
m	Molecular mass
Nu	$(=h'W'/k')$ Nusselt number - dimensionless
P'	Pressure
Pr	$(=\mu' c'_p/k')$ Prandtl number - dimensionless
R	Gas constant
Ra	$(=(T'_h - T'_c)\bar{\beta}'W'^3 g'_o/\alpha'\nu')$ Rayleigh number - dimensionless
T	$(=(T' - T'_c)/(T'_h - T'_c))$ Dimensionless temperature
T'_h	Temperature of hot plate
T'_c	Temperature of cold plate

T'	Temperature at any point in the fluid
$\bar{T}$	Mean temperature
$\Delta T'$	Temperature difference = $T'_h - T'_c$
t	$(=\nu' t'/W'^2)$ Dimensionless time
t'	Time
u'	x' component of velocity
u	x component of velocity - dimensionless
V	$(=V'/a'\omega')$ Dimensionless enclosure motion
V'	Enclosure motion
V_s	Velocity of sound
v'	y' component of velocity
v	y component of velocity - dimensionless or particle velocity
W'	Enclosure width
x	Cartesian coordinate or dimensionless coordinate $(=x'/W')$
x'	Cartesian coordinate
y	$(=y'/w')$ Dimensionless coordinate
y'	Cartesian coordinate
Z	Acceleration level
α'	Thermal diffusivity
$\bar{\beta}'$	Volumetric coefficient of thermal expansion
λ	Wave length
μ'	Dynamic viscosity
ν'	$(=\mu'/\rho')$ Kinematic viscosity
ρ'	Density
$\bar{\rho}'$	Mean density
θ	Angle

ω'	$(=2\pi f)$ Frequency
Ψ	$(=\Psi'/\alpha')$ Dimensionless stream function
Ψ'	Stream function
ζ'	$(=-\nabla^2\Psi')$ Vorticity
ζ	$(=\frac{\zeta' W'^2}{\alpha'})$ Dimensionless vorticity

SUBSCRIPTS

c	Vertical boundary at $x' = W'$
h	Vertical boundary at $x' = 0$
p	Constant pressure
v	Constant volume
x	x direction
y	y direction
o	Before collision

SUPERSCRIPTS

(1)	Molecule 1
(2)	Molecule 2
*	Value of dependent variable after time $\Delta t/2$
'	Value of dependent variable after time Δt.

ABSTRACT

This report describes the results of an investigation of the effect of vibration on convective heat transfer. The research was divided into three areas; an analytical solution was sought, an experimental apparatus was constructed, and a theoretical investigation was conducted into the effect of vibration on the properties of a cryogenic fluid.

<u>Analytical</u>: A mathematical model of the following problem is derived:

> Consider the laminar two-dimensional natural convection of a fluid enclosed between two plane parallel vertical boundaries which are held at different uniform temperatures. The space between the vertical boundaries is closed by two insulated horizontal boundaries. In the remaining direction the space is considered to extend to infinity. The enclosure is subjected to vibration in a direction parallel to or perpendicular to the gravity field.

In the derivation of the governing equations of the above problem the enclosed fluid under thermal equilibrium conditions is assumed to move as a bulk when the enclosure is vibrated. The validity of this assumption, which is crucial to the mathematical formulation, is discussed in detail and it is shown that the anticipated range of enclosure vibratory conditions over which the assumption should be valid is not negligibly small. The governing equations are solved using numerical techniques and an IBM 360 digital computer. The computer program is verified by comparing computational results for the non-vibratory case with those results currently available in the literature. Nusselt number versus time curves for two enclosure vibratory conditions for air as the enclosed fluid are presented and compared to the curve for the stationary enclosure case.

The problem is also analyzed using the technique of dimensional analysis,

and a complete and independent set of dimensionless parameters is derived. This analysis considers the total problem without employing simplifying assumptions.

Experimental: Subsequent to the decision to study a rectangular geometry, design criteria were established for the construction of a test cell. These criteria involved the characteristics of the electrodynamic vibration system which was used. A preliminary dummy cell was constructed to investigate the characteristics of induced transverse vibrations. The design of the test cell then proceeded with due regard for the "crosstalk" problem as well as the necessity for extensive thermal and vibration instrumentation. A guarded electrical hot plate was used in conjunction with a water or refrigerant cooled cold plate. The test cell may be expanded to several widths by the insertion of various plexiglass sidewall assemblies.

The test cell was constructed and fully instrumented and subjected to a series of validating non-vibratory tests. Results of these tests are shown which indicate that the apparatus will yield high integrity data.

Transport Properties: An effort was made to determine the effect of vibrations upon the transport properties of liquids from the molecular theory of matter. It was concluded from a study of the literature that the molecular theory of liquids has not reached the stage of development where analytical methods may be utilized to determine the transport properties even under static conditions. As a result of this conclusion, attention was turned to the molecular theory of gases and solids. The idea here was that, although the primary interest in this project is in cryogenic liquids, if one could determine that vibration had some effect upon the properties of either gases or solids, then one could deduce at least some sort of order of magnitude of the effect of vibrations upon the transport properties of liquids. Again, however, the theory of solids and gases is not well enough developed to allow quantitative calculation of the effect of vibrations upon the transport properties. It was concluded that any effort dealing with the effect of vibrations upon transport properties of matter should be approached from the experimental standpoint rather than from the analytical standpoint.

In dealing with the effect of vibrations upon heat transfer the question arises as to whether or not the vibrations contribute significantly to the energy input to the system under study. If the vibrational energy input to the material is significant, the energy equations must be modified to account for this additional energy input. Molecular theory was used to evaluate the energy input to a system composed of an ideal gas under the influence of vibrations. It was possible to evaluate time rate of energy input due to vibrations of an ideal gas confined within a rectangular enclosure. Equations were derived for the time rate of energy input as a function of acceleration level, vibrational frequency and container dimensions for ideal gases. In addition, equations were derived for determining the equilibrium temperature which would result when an ideal gas confined within a rectangular enclosure was subjected to vibrations at a particular level over some interval of time. Energy inputs and equilibrium temperatures were evaluated for example cases.

INTRODUCTION

It is the purpose of this research program to determine the effects of different vibrational modes on convective heat transfer in enclosures. The need for information of this nature is broad but has been pointed to in one instance by design engineers concerned with the effect of vibration on heat transfer in such cryogenic systems as rocket propellant tanks and transfer systems. The aim of this research effort is to develop a correlation equation for use by design engineers to predict heat flux variation in enclosures due to vibration.

The first step taken to accomplish this task was to conduct a complete literature search concerning the effect of vibration on natural convective heat transfer. It soon became obvious that the effort could be naturally divided into three distinct areas of study. The first would be an analytical investigation of the general problem of the effect of vibration on convective heat transfer in an enclosure. The second would be the conduction of an experimental program to validate predictions of the analytical study and, as well, to provide data where none presently exists. The third area would be an attempt to determine what effect, if any, intense vibration has on the transport properties of a cryogenic fluid.

The first year's research effort was then directed in these three parallel paths. The first task involved a literature survey in which 225 articles were catalogued as being of possible interest to this research. A number of these articles which were considered most pertinent to the specific problem being investigated were abstracted for quick reference to their content. These abstracts were presented in the First Quarterly Report. The bibliography is included as an appendix to this report for completeness. It is broken into two sections: first, a section concerning natural convective heat transfer and the effect of vibrations on such heat transfer, and second, a section concerning the effect of vibrations on transport properties of fluids.

A study of those papers listed in the first section reveals that three categorizations can be made. The first would include those papers dealing with the analytical prediction of heat transfer in enclosures without vibrations, the second would include those papers dealing with experimental determination of convective heat transfer rates in enclosures without vibrations, and the third would be those papers concerning the effect of vibration on heat transfer rates from any configuration. Although there is a wealth of information available in these three categories, it was found that there was no published information concerning either analytical or experimental determination of convective heat transfer rates within enclosures subjected to vibratory stresses. It was thus felt that fundamental work was necessary in order to produce data and techniques which would allow the prediction of convective heat transfer rates in enclosures subjected to different modes of vibration. Because of this, and for the sake of simplicity, rectangular enclosures were chosen for study. Such a geometry would allow for both analytical and experimental study of a one dimensional heat flux problem in which the heat transfer surfaces could either be made isothermal or constant flux and, as well, the vibration vector and heat flux vector could be oriented perpendicular or parallel to each other. Once this decision was made an analytical approach was begun immediately and the design of a test apparatus was also started. The work which followed is described in the sections below.

The review of literature concerning the effect of vibration on the transport properties of a fluid revealed that no information was available on this specific topic. As a result a fundamental analytical study was initiated in an attempt to determine if a theory could be evolved which would predict whether or not intense mechanical vibrations would affect the transport properties of a fluid. A description of these efforts is provided in the following section.

Thus the needs of design engineers concerned with a very practical problem have pointed to a very broad area of technology for which there exists a distinct lack of knowledge of phenomena and absence of fundamental data. The following sections will describe a three-pronged approach used to develop techniques, data, theories, and predictive power in this area of the effect of vibrations on convective heat transfer in enclosures.

ANALYSIS OF PROGRESS

ANALYTICAL

The Natural Convection Problem and Its Formulation

Consider the laminar two-dimensional natural convection of a fluid enclosed between two plane parallel vertical boundaries a distance W' apart which are held at different temperatures. The space between the vertical boundaries is closed by two horizontal boundaries a distance H' apart where $H' \gg W'$ (see Fig. 1). In the remaining direction, at right angles to the plane of the sketch in Figure 1, the space is considered to extend to infinity. The enclosure, formed as described above and depicted in Figure 1, is subjected to either longitudinal or transverse vibration.

The problem is now formulated with respect to a moving co-ordinate system which is fixed to the vibrating enclosure. Under vibratory conditions, the enclosure and confined fluid are assumed to vibrate together as a bulk, i. e., no relative motion exists between the enclosure walls and the confined liquid as a result of the vibration. This assumption is crucial in that the mathematical formulation of the problem depends in large measure upon how the vibration effects are incorporated into the momentum equations. Additional assumptions relating to the manner in which density variations are considered are also required, and these assumptions must be weighed quite carefully in the final analysis. Each of the aforementioned assumptions will be discussed and justified at the point in the development where it is made.

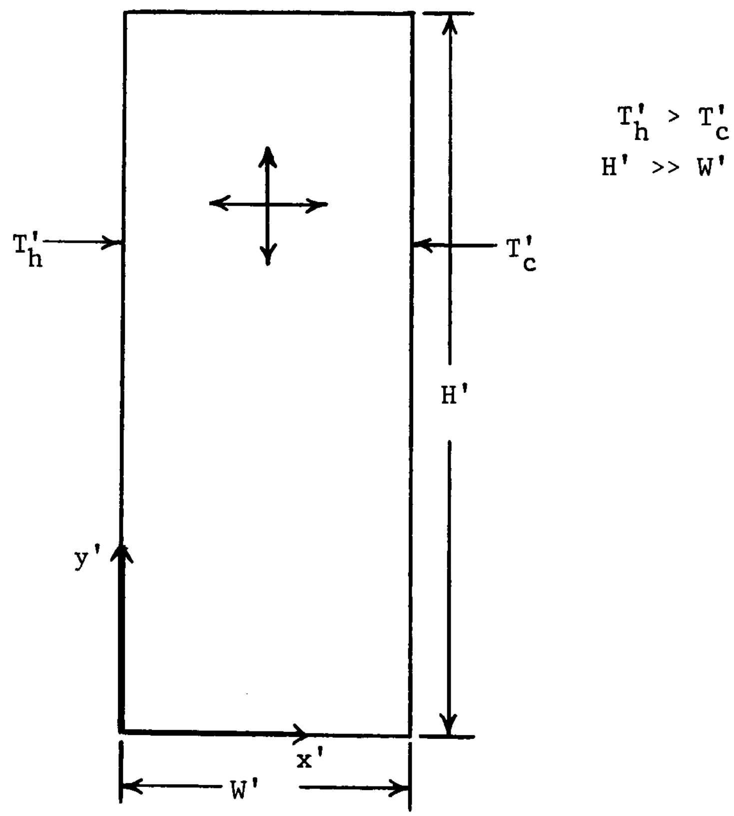

Figure 1. Schematic Diagram of Rectangular Enclosure Subjected to Either Transverse or Longitudinal Oscillations.

Let T_h' and T_c' be the temperatures of the left and right vertical boundaries respectively. (All dimensional quantities are primed and all dimensionless quantities are unprimed.) Since pressure differences in the fluid will be small in comparison to the absolute pressure, variations in density will be determined by variations in the temperature T'. If the ratio $(T_h' - T_c')/T_h'$ is small, variation in the temperature of the fluid normally needs to be considered only in the determination of the buoyancy force. However, since there is an added force in the momentum equations due to the enclosure motion, which involves the density of the fluid, and since it is of primary importance to determine the coupling of these two forces, variations in the density will also be considered in this added d'Alembert force. In the other force terms of the momentum equations the density will be considered uniform at its mean value, i. e., its value at $\overline{T} = (T'_h + T'_c)/2$.

Considering the assumptions described above, the equation of conservation of mass is

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{1}$$

where x', y' are the coordinates as shown in Figure 1 and the corresponding velocities are u', v'.

Assuming no temperature change in the fluid due to compression and/or viscous dissipation, the energy equation becomes

$$\frac{1}{\alpha'} \frac{DT'}{Dt'} = \nabla^2 T' \quad . \tag{2}$$

In equation (2) $\frac{D}{Dt'}$ is the substantial derivative and ∇^2 is the Laplacian operator and are given by

$$\frac{D}{Dt'} = \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}$$

$$\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \qquad \text{, and}$$

t' is time and α' is the thermal diffusivity ($k'/\bar{\rho}'c'_p$ for gases, $k'/\bar{\rho}'c'_v$ for liquids where $\bar{\rho}'$ is the density evaluated at $\bar{T}'$, k' is thermal conductivity of the fluid and c'_p, c'_v are the specific heats); α' is considered uniform at its value at $\bar{T}'$.

Considering transverse vibration (x' direction) the momentum equations are:

$$\frac{Du'}{Dt'} = -\frac{1}{\bar{\rho}'}\frac{\partial P'}{\partial x'} + \nu'\nabla^2 u' - \frac{\rho'}{\bar{\rho}'}\frac{dV'}{dt'} \qquad (3)$$

$$\frac{Dv'}{Dt'} = -\frac{1}{\bar{\rho}'}\frac{\partial P'}{\partial y'} + \nu'\nabla^2 v' + \frac{B_y'}{\bar{\rho}'} \qquad (4)$$

where P' is the pressure in the fluid, ν' is the kinematic viscosity $(= \frac{\mu'}{\rho'})$, V' is the enclosure motion, and B_y' is the body force. B_y' is equal to $-\rho' g_o'$ which, considering the previous assumption regarding the smallness of $(T_h' - T_c')/T_h'$, may be written

$$B_y' = -\bar{\rho}' g_o' \bar{\beta}' T' + C \qquad (5)$$

where $\bar{\beta}'$ is the volumetric coefficient of thermal expansion, g_o' is the acceleration of gravity, and C is a constant which is of no significance in that it falls out of the final equations. The enclosure motion is assumed sinusoidal so that

$$V' = a'\omega' \cos \omega' t' \qquad (6)$$

where a' and ω' are the amplitude and frequency of vibration respectively. The last term in equation (3) is a "so-called" d'Alembert force which arises from the vibration of the enclosure and the assumption that the enclosure and confined fluid move as a bulk.

The governing equations may be simplified by introducing the stream function which satisfies the continuity equation identically. The velocity components are then given in terms of the stream function, Ψ', as:

$$u' = \frac{\partial \Psi'}{\partial y'} \qquad \text{and}$$

$$v' = -\frac{\partial \Psi'}{\partial x'} \qquad .$$

It is also convenient to introduce the vorticity, ζ', defined by the following equation:

$$\nabla^2 \Psi' = -\zeta' \tag{7}$$

The governing equations may be put in dimensionless form by defining the following dimensionless variables:

$$\Psi = \frac{\Psi'}{\alpha'} \quad , \quad \zeta = \frac{\zeta' W'^2}{\alpha'} \ ,$$

$$T = \frac{T' - T_c'}{T_h' - T_c'} \quad , \quad V = \frac{V'}{a'\omega'} \quad , \quad t = \frac{\nu' t'}{W'^2} \quad ,$$

$$x = \frac{x'}{W'} \quad , \quad \text{and} \quad y = \frac{y'}{W'} \quad .$$

The energy equation (2) may now be written as

$$Pr \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \nabla^2 T \tag{8}$$

$$\text{where } u = \frac{\partial \Psi}{\partial y} \tag{9}$$

$$v = -\frac{\partial \Psi}{\partial x} \quad \text{and} \tag{10}$$

Pr is the Prandtl number $\left(\frac{\mu' c_p'}{k'}\right)$.

The vorticity equation becomes:

$$\nabla^2 \Psi = -\zeta \quad . \tag{11}$$

The pressure is now eliminated from equations (3) and (4), ρ' is eliminated with the use of equation (5), and equation (6) is used in writing the d'Alembert force. These operations give

$$\frac{\partial\zeta}{\partial t} + \frac{u}{Pr}\frac{\partial\zeta}{\partial x} + \frac{v}{Pr}\frac{\partial\zeta}{\partial y} = \nabla^2\zeta$$

$$+Ra\left[\frac{\partial T}{\partial x} + \frac{g'}{g'_o}\sin\frac{\omega' W'^2 t}{\nu'}\frac{\partial T}{\partial y}\right] \qquad (12)$$

where Ra is the Raleigh number $= \frac{(T'_h - T'_c)\beta' W'^3 g'_o}{\alpha'\nu'}$ and $g' = a'\omega'^2$ is the maximum acceleration of the enclosure. For longitudinal vibrations, y' direction, equation (12) needs only to be modified by changing the sign of the last term in braces on the right-hand side to minus and the $\frac{\partial T}{\partial y}$ to $\frac{\partial T}{\partial x}$. The remaining equations are unaltered.

The initial conditions are:

$$\Psi(x,y,o) = o$$

$$u(x,y,o) = o$$

$$v(x,y,o) = o$$

$$\zeta(x,y,o) = o \qquad \text{and}$$

$$T(x,y,o) = o \qquad \text{(uniform initial temperature).}$$

The appropriate boundary conditions are:

$$\Psi(o,y,t) = \Psi(1,y,t) = o\ ,\ T(o,y,t) = 1$$

$$\Psi(x,o,t) = \Psi(x,\tfrac{H'}{W'},t) = o,\ T(1,y,t) = o$$

$$u(x,o,t) = u(x,\tfrac{H'}{W'},t) = o$$

$$v(o,y,t) = v(1,y,t) = o \quad \text{and}$$

$$\frac{\partial T(x,o,t)}{\partial y} = \frac{\partial T(x,\frac{H'}{W'},t)}{\partial y} = o$$

The fluid motion within the enclosure, and thus the heat transfer rate, is governed by equations (8), (9), (10), (11), and (12) along with the initial and boundary conditions.

Compressibility Effects

The mathematical formulation of the problem of predicting the effects of vibration upon natural convection between two vertical parallel isothermal surfaces was presented in the preceeding paragraphs. In the formulation it was assumed, as has been done by other researchers studying the effects of vibration on natural convection from single plane surfaces (154), that the confined fluid and the enclosing surfaces move together as a bulk. This assumption permitted modification of the momentum equations to include implicitly a force term due to the vibration. The validity of this assumption is questionable over wide frequency and amplitude ranges; however, it has been shown (154) to yield reasonable results at relatively low maximum enclosure velocities.

Since all real fluids are compressible to some extent, there will be points within the confined fluid where the velocity, acceleration, and displacement are not those of the enclosure itself. The extent of this difference would be difficult to estimate theoretically to any degree of accuracy; however, it should be possible to establish an estimate of an upper bound where this difference is negligible. If one considers a single plane surface vibrating in a fluid of infinite extent, it may be shown (154) that if the maximum surface velocity is small in comparison to the velocity of sound in the fluid, compressibility due to the vibration should not be important. This conclusion was reached by considering (1) the wall to be the source of a sound wave with a wave length normal to the wall $\lambda = \frac{V_s}{2\pi f}$ where V_s is the velocity of sound in the fluid and f is the frequency of vibration; (2) the maximum compression occurring over half this wave length to be 2a', where a' is the amplitude of wall vibration; and (3) considering the ratio $\frac{4a'}{\lambda} = \frac{8a'\pi f}{V_s}$ to be the relative compressibility (reference (154) has $\frac{4a'}{\lambda} = \frac{2a'f}{\pi V_s}$ which is incorrect). For the example quoted in (154) the relative compressibility should have been 0.00552 inch/inch instead of 0.001 inch/inch.

For the specific case of a fluid confined in a finite enclosure undergoing vibrations, additional factors must be considered, i. e., wave reflections from the enclosing walls. In this case the maximum compression that could occur in one-half wave length is 4a' instead of 2a' since a standing wave with a 2a' maximum amplitude could be generated. Thus the relative compressibility, compression per unit distance, becomes

$$C \equiv \frac{8a'}{\lambda} = \frac{16a'f\pi}{V_s} \quad . \tag{13}$$

If we assume simple harmonic oscillation of the enclosure, the maximum enclosure amplitude a' is given by

$$a' = \frac{ag}{(2\pi f)^2} \quad , \tag{14}$$

where g is the acceleration of gravity and a is the enclosure acceleration in g's. The maximum enclosure velocity V is defined by

$$V = 2\pi a'f \quad . \tag{15}$$

To simplify the graphical presentation of equations (13), (14), and

(15) for water as the enclosed fluid, both sides of each of these equations are divided by a, equation (14) is combined with equations (13) and (15), and V_s is set equal to 5000 fps to give

$$a'/a = 9.8/f^2 \qquad \text{(in)}, \qquad (16)$$

$$C/a = 492/f \qquad \text{(in/in), and} \qquad (17)$$

$$V/a = 61.6/f \qquad \text{(in/sec)} \quad . \qquad (18)$$

(In the above equations f has the units cycle per second.)

Table 1 gives computed values of a'/a, C/a, and V/a for various values of the frequency of oscillation between 100 and 3000 cps, and Figure 2 shows a plot of these equations. To use this figure one enters the plot with a given value of f and reads the values of a'/a, C/a, and V/a from the curves; when these quantities are multiplied by selected values of the acceleration and then the maximum amplitude a', the maximum enclosure velocity V and the relative compressibility C are obtained.

Table 2 gives values of the relative compressibility C with water as the enclosed fluid for various maximum enclosure velocities (values of f, a', and a are also given). For $V < 0.616$ in/sec we see that $C < 0.000984$ in/in meaning that there is approximately 0.1% compression, and the assumption of bulk liquid motion should be valid for maximum enclosure velocities less than this value. However, it should be emphasized that the equation for the determination of the relative compressibility C should give only a rough estimate of the degree of compression, and predictions based upon it should be verified by experiment.

Since the range of maximum enclosure velocity over which the "bulk-motion" assumption is considered valid is not negligibly small, solution

TABLE 1. Tabulation of a'/a, C/a, and V/a for Various Values of Frequency f.

f cps	$(C/a) \times 10^6$ inch/inch	$(V/a) \times 10^4$ inch/second	$(a'/a) \times 10^8$ inch
100	984.0	6160	98,000
200	492.0	3080	24,500
300	328.0	2050	10,920
400	246.0	1540	6,120
500	196.3	1233	3,930
600	163.6	1027	2,730
700	140.4	882	2,000
800	123.0	770	1,534
900	109.5	686	1,210
1000	98.4	616	980
1100	89.7	560	811
1200	81.8	513	683
1300	75.6	473	580
1400	70.2	440	499
1500	65.6	411	435
1600	61.3	385	383
1700	57.8	363	340
1800	54.8	342	303
1900	51.8	324	272
2000	49.2	308	245
2100	46.8	293	222
2200	44.6	280	203
2300	42.8	268	185
2400	41.0	257	170
2500	39.4	246	157
2600	37.9	237	145
2700	36.4	229	135
2800	35.1	220	125
2900	33.9	213	116
3000	32.8	205	109

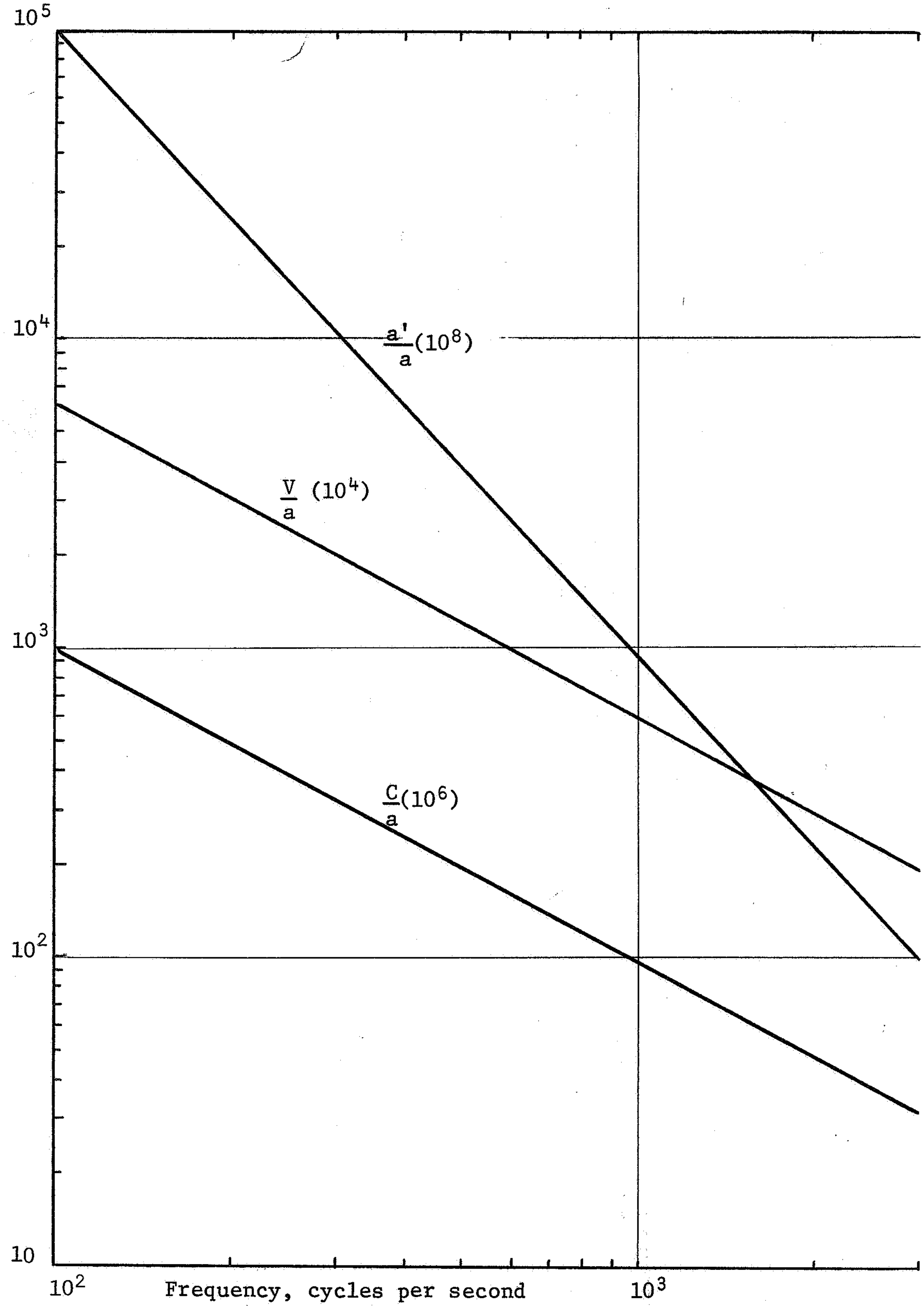

Figure 2. Normalized Maximum Amplitude, Enclosure Velocity, and Compressibility.

TABLE 2. Tabulation of a', C, and V for Various Accelerations a and Frequencies f.

f cps	$C \times 10^6$ inch/inch	$V \times 10^4$ inch/second	$a' \times 10^8$ inch
		a=5	
100	4920	30,800	490,000
500	982	6,170	19,660
1000	492	3,080	4,900
2000	246	1,540	1,225
3000	164	1,025	545
		a=10	
100	9840	61,600	980,000
500	1963	12,330	39,300
1000	984	6,160	9,800
2000	492	3,080	2,450
3000	328	2,050	1,090
		a=15	
100	14,760	92,500	1,470,000
500	2,950	18,500	59,000
1000	1,476	9,250	14,700
2000	738	4,620	3,670
3000	492	3,070	1,635
		a=20	
100	19,680	123,200	1,960,000
500	3,926	24,660	78,600
1000	1,968	12,320	19,600
2000	984	6,160	4,900
3000	656	4,100	2,180

Table 2. (continued)

f cps	C x 10^6 inch/inch	V x 10^4 inch/second	a' x 10^8 inch
		a=25	
100	24,600	154,000	2,450,000
500	4,910	30,800	98,200
1000	2,460	15,400	24,500
2000	1,230	7,700	6,120
3000	820	5,120	2,730

of the mathematical model (equations (8), (9), (10), (11), and (12) including initial and boundary conditions) should yield useful information concerning the fluid motion within the enclosure and the heat transfer rates.

However, due to the complex nature of the governing differential equations, attempts at a closed-form solution would be quite time consuming with very little chance of obtaining a solution valid over a significant range of the pertinent variables. Thus, a numerical solution was undertaken. It is believed that more useful analytically based results can be obtained in a much shorter period of time by the numerical method.

Wilkes and Churchill (181) present the results of a numerical solution of the problem of natural convection in a rectangular enclosure without vibrations; details of the numerical procedures are contained in a Ph.D. thesis by Wilkes (180).

Finite Difference Formulation of the Natural Convection Problem

In obtaining a finite difference approximation to a partial differential equation it is first necessary to establish a system of grid points. For the following work the grid system shown in Figure 3 will be used.

The implicit alternating direction method discussed by Wilkes (180) will be used in formulating the finite difference equations. In using the implicit alternating direction method one considers two successive half time steps $\Delta t/2$. During the first half time step all x space derivatives are approximated implicitly and all y space derivatives are approximated explicitly. During the second half time step the procedure is reversed and all x derivatives are expressed explicitly and y derivatives are expressed implicitly. The procedure is optional however, and, if one desires, x derivatives may be expressed explicitly and y derivatives implicitly during the first half time step and the procedure again reversed during the second half time step.

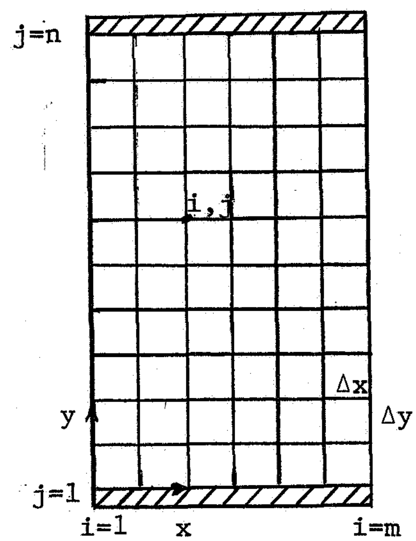

Figure 3. Grid System.

All derivatives will be approximated using central difference forms whenever possible because of the resultant higher order truncation error as opposed to forward or backward differences.

Considering the finite difference approximation for the energy equation during the first half time step, using central differences,

$$\Pr\left[\frac{T^*_{i,j} - T_{i,j}}{\Delta t/2}\right] + u_{i,j}\left[\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x}\right] + v_{i,j}\left[\frac{T^*_{i,j+1} - T^*_{i,j-1}}{2\Delta y}\right] = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} + \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta y)^2} \qquad (19)$$

where x space derivatives have been expressed explicitly and y space derivatives have been expressed implicitly. The superscript * will be used hereinafter to indicate the value of a dependent variable after a time increment $\Delta t/2$. The velocity terms u and v have been held constant during the half time step at their "old" value. During the second half time step:

$$Pr\left[\frac{T'_{i,j} - T^*_{i,j}}{\Delta t/2}\right] + u_{i,j}\left[\frac{T'_{i+1,j} - T'_{i-1,j}}{2\Delta x}\right] + v_{i,j}\left[\frac{T^*_{i,j+1} - T^*_{i,j-1}}{2\Delta y}\right] =$$

$$\frac{T'_{i-1,j} - 2T'_{i,j} + T'_{i+1,j}}{(\Delta x)^2} + \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta y)^2} \quad (20)$$

where x space derivatives have been expressed implicitly and y space derivatives have been expressed explicitly, again using central differences. The superscript ' will be used to designate the value of a dependent variable after a second half time step $\Delta t/2$ or after a total time Δt. The variables u and v have again been held constant during the second half time step.

Solving equation (19) for T* in terms of T:

$$T^*_{i,j-1}\left[-\frac{v_{i,j}}{2\Delta y} - \frac{1}{(\Delta y)^2}\right] + T^*_{i,j}\left[\frac{2Pr}{\Delta t} + \frac{2}{(\Delta y)^2}\right]$$

$$+ T^*_{i,j+1}\left[\frac{v_{i,j}}{2\Delta y} - \frac{1}{(\Delta y)^2}\right] = T_{i-1,j}\left[\frac{u_{i,j}}{2\Delta x} + \frac{1}{(\Delta x)^2}\right]$$

$$+ T_{i,j}\left[\frac{2Pr}{\Delta t} - \frac{2}{(\Delta x)^2}\right] + T_{i+1,j}\left[-\frac{u_{i,j}}{2\Delta x} + \frac{1}{(\Delta x)^2}\right] \quad (21)$$

which holds for all interior grid points such that

$$2 \leqslant i \leqslant m - 1$$

$$2 \leqslant j \leqslant n - 1 \quad .$$

Considering j = 1, or along the bottom insulated wall: $u_{i,1} - 0$ and

$$\left[\frac{\partial T^*}{\partial y}\right]_{i,1} = 0 \qquad \text{thus} \quad Pr\,\frac{\partial T}{\partial t} = \nabla^2 T$$

Considering a Taylor's series expansion about $T^*_{i,1}$ we have

$$T^*_{i,2} = T^*_{i,1} + \Delta y \left[\frac{\partial T^*}{\partial y}\right]_{i,1} + \frac{(\Delta y)^2}{2}\left[\frac{\partial^2 T^*}{\partial y^2}\right]_{i,1} + \ldots$$

thus

$$\left[\frac{\partial^2 T^*}{\partial y^2}\right]_{i,1} = \frac{2(T^*_{i,2} - T^*_{i,1})}{(\Delta y)^2} \quad .$$

Rewriting equation (19) for the special case j = 1

$$Pr\left[\frac{T^*_{i,1} - T_{i,1}}{\Delta t/2}\right] = \frac{T_{i-1,1} - 2T_{i,1} + T_{i+1,1}}{(\Delta x)^2} + \frac{2(T^*_{i,2} - T^*_{i,1})}{(\Delta y)^2} \quad .$$

Solving for T^* in terms of T

$$T^*_{i,1}\left[\frac{2Pr}{\Delta t} + \frac{2}{(\Delta y)^2}\right] + T^*_{i,2}\left[-\frac{2}{(\Delta y)^2}\right] = \frac{(T_{i-1,1} + T_{i+1,1})}{(\Delta x)^2} + T_{i,1}\left[\frac{2Pr}{\Delta t} - \frac{2}{(\Delta x)^2}\right] \qquad (22)$$

which holds for grid points such that

$$2 \leqslant i \leqslant m - 1$$

$$j = 1 .$$

Similarly along the top insulated wall where $j = n$

$$T^*_{i,n-1} \left[- \frac{2}{(\Delta y)^2} \right] + T^*_{i,n} \left[\frac{2Pr}{\Delta t} + \frac{2}{(\Delta y)^2} \right] =$$

$$\frac{(T_{i-1,n} + T_{i+1,n})}{(\Delta x)^2} + T_{i,n} \left[\frac{2Pr}{\Delta t} - \frac{2}{(\Delta x)^2} \right] \qquad (23)$$

which holds for grid points such that

$$2 \leqslant i \leqslant m - 1$$

$$j = n \quad .$$

Thus equations (21), (22), and (23) give the new values of temperature T^*, after a time $\Delta t/2$, in terms of the old temperature T, Pr, u, v, Δx, Δy, and Δt. The matrix of this system has zeros everywhere except on the main diagonal and on the two diagonals parallel to and on either side of the main diagonal. Such a system is called a tridiagonal system and may be solved directly using a simple iterative method discussed by Wilkes (180) and Forsythe (64). Subsequent consideration of the energy and vorticity equations in this report will always lead to a tridiagonal system and it is for this very reason that the implicit alternating direction method is utilized.

A similar consideration may be made for the energy equation during the second half time step giving values of T' in terms of T^*, Pr, u, v, Δx, Δy, and Δt. Details of this development and the development of the complete set of equations used in the finite difference computations is given in Appendix I. The specific computational procedures to be utilized in solving the system of partial differential equations is also given in Appendix I. Generally the procedures are:

(1) Solve the tridiagonal system to determine values of temperature at the advanced point in time,

(2) Solve the tridiagonal system to determine values of interior vorticity at the advanced point in time,

(3) Solve the tridiagonal system to determine values of the stream function at the advanced point in time,

(4) Using new values of stream function, compute new values for the velocity components u and v,

(5) Compute new values for vorticity on the enclosure boundaries,

(6) Compute the mean Nusselt number.

The sequence (1) through (6) is applied repeatedly until a steady state condition is reached.

Although the associated problem formulation and programming using an implicit alternating direction method is more complicated than for either an implicit or explicit method alone, there are two main advantages of this method. These advantages are:

(1) The m by n tridiagonal matrix system is more easily solved than is the general m by n system arising in the straight explicit or implicit method.

(2) Larger time increments may be utilized in the implicit alternating direction method as compared to the explicit or implicit method, thus allowing a considerable saving

in computer time.

As a check on the finite difference formulation of the governing equations and associated computer programming, several runs were made considering the non-vibratory enclosure problem. Results of the non-vibratory problem compared favorably to those obtained by Wilkes (180) in all cases tested.

Computations have also been made for two enclosure vibratory conditions; (1) an acceleration of 1g and a frequency of 100 cps and (2) an acceleration of 5g's and a frequency of 100 cps. Figure 4 is a plot of Nusselt number versus non-dimensional time for a square enclosure filled with air. For these two vibratory conditions and for the stationary case, all three curves are for a Grashof number of 10^4. The properties of the air are evaluated at a bulk temperature of 100^oF. Since the enclosure width occurs as a parameter in the vibratory case, the width W' is assumed to be 10 inches.

At t = 0.12 the Nusselt number for the stationary case has reached its steady state value. For the 1g acceleration case, the Nusselt number versus t curve differs slightly from the curve for the stationary problem which indicates negligible vibration effect at this acceleration level. There is a noticeable effect in the Nusselt number versus t curve as the acceleration level of the enclosure is increased from the 1g to the 5g level. The 1g and 5g's curves appear to be sinusoidal about the steady state curve for the stationary case. This observation can not be substantiated until additional computations have been completed.

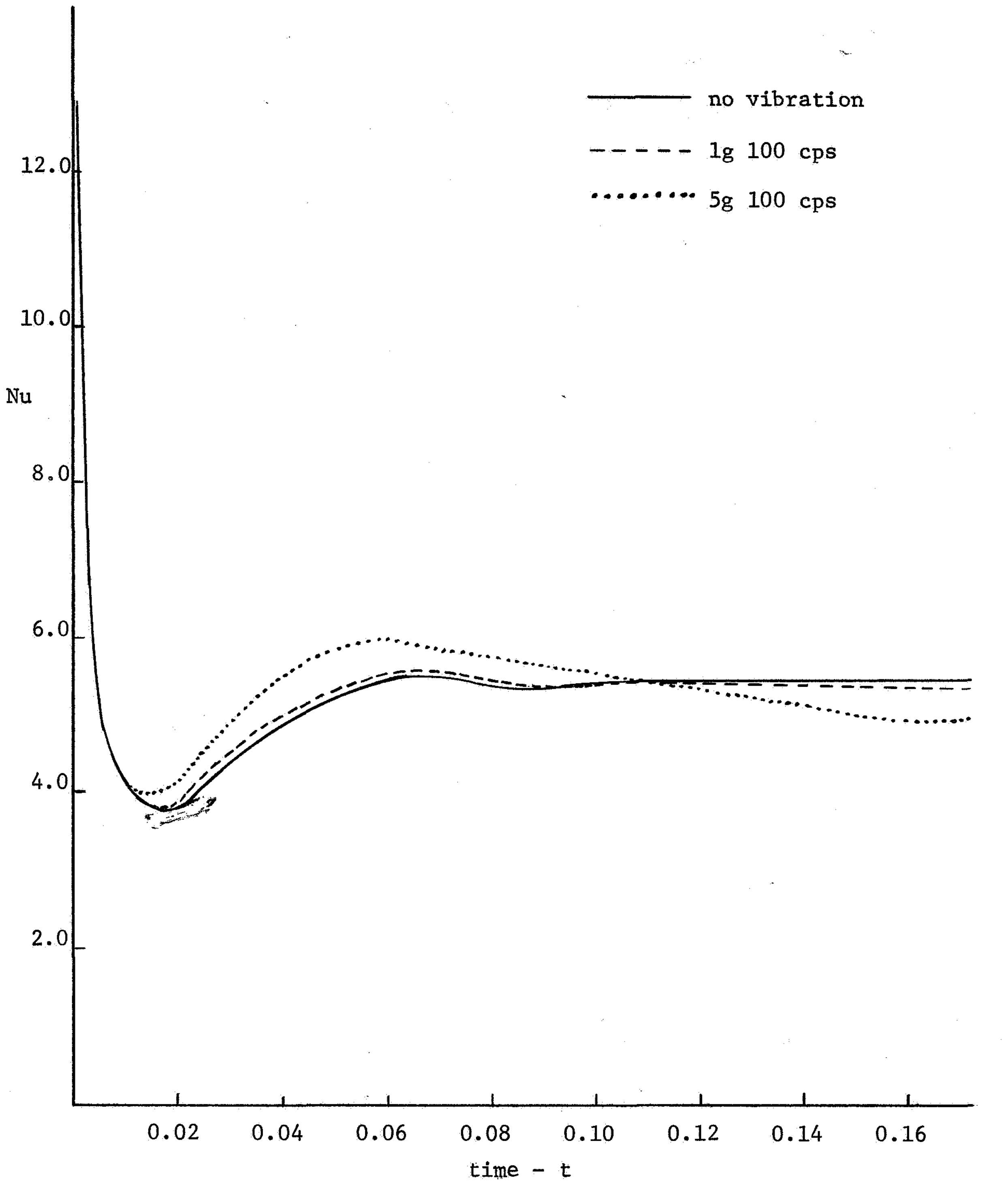

Figure 4. Nusselt Number Versus Time for an Air-Filled Square Enclosure Subjected to Vibratory Motion.

Dimensional Analysis

Consider the natural convection of a fluid enclosed between two plane parallel vertical boundaries a distance W' apart which are held at different temperatures. The space between the vertical boundaries is closed by two horizontal boundaries a distance H' apart, where H' > W'. The enclosure is subjected to either longitudinal or traverse vibration. It is desired to determine the dimensionless parameters which are pertinent to this problem in order to establish a basis for the experimental work and to properly interpret and correlate the experimental results.

A set of dimensionless parameters may be determined by normalizing the equations governing free convection in a vibrating enclosure. Alternately, a more general set of parameters may be determined by performing a dimensional analysis of the problem without simplifying assumptions using those variables which are considered pertinent. The parameters will now be obtained using the method of dimensional analysis.

The first and most important step in performing a dimensional analysis of a problem is to construct a list of variables. For the problem under consideration here, the list of variables should contain those variables which are pertinent to the problem in the absence of vibration plus those variables which are considered necessary to incorporate the vibration effects.

The following variables have been found by others to be required for the non-vibratory problem:

h' - convective heat transfer coefficient

$\Delta T' = (T_h' - T_c')$ temperature difference

W' - width of enclosure

H' - height of enclosure

$\overline{\rho}'$ - fluid density

μ' - fluid dynamic viscosity

k' - fluid thermal conductivity

$\overline{\beta}'$ - volumetric coefficient of thermal expansion

g_o' - acceleration of gravity

c_p' - fluid specific heat at constant pressure

As a minimum, the following additional variables will be required to account for the vibration (the vibration is assumed sinusoidal):

a' - maximum amplitude of enclosure vibration

ω - frequency of vibration

V_s - speed of sound

Forming a dimensional matrix of the variables we have:

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	k_{13}
	h'	$\Delta T'$	H'	$\overline{\rho}'$	V_s'	g_o'	c_p'	a'	$\overline{\beta}'$	W'	μ'	k'	ω'
Q'	1	0	0	0	0	0	1	0	0	0	0	1	0
M'	0	0	0	1	0	0	-1	0	0	0	1	0	0
L'	-2	0	1	-3	1	1	0	1	0	1	-1	-1	0
T'	-1	0	0	0	-1	-2	0	0	0	0	-1	-1	-1
θ'	-1	1	0	0	0	0	-1	0	-1	0	0	-1	0

$$\text{Since} \quad \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{vmatrix} \neq 0,$$

the rank of the dimensional matrix is five, and thus there are 13 - 5 = 8 dimensionless parameters required to form a complete set. In addition, g_o'

and β' appear in the governing equations only as a product and will thus appear as a product in the dimensionless parameters.

Writing the equations for the exponents k (j = 1, 2,....., 13), we obtain:

$$
\begin{array}{lllllllllllllll}
k_1 & & & & & & +k_7 & & & & & +k_{12} & & = 0 & (24) \\
 & & & k_4 & & & -k_7 & & & & +k_{11} & & & = 0 & (25) \\
-2k_1 & & +k_3 & -3k_4 & +k_5 & +k_6 & & +k_8 & & +k_{10} & -k_{11} & -k_{12} & & = 0 & (26) \\
-k_1 & & & & -k_5 & -2k_6 & & & & & -k_{11} & -k_{12} & -k_{13} & = 0 & (27) \\
-k_1 & +k_2 & & & & & -k_7 & & -k_9 & & & -k_{12} & & = 0\,. & (28)
\end{array}
$$

Equations (24) through (28) are solved for k_9 through k_{13} to give:

$$k_9 = k_2 \qquad (29)$$
$$k_{10} = k_1 - k_3 + 2k_4 - k_5 - k_6 - k_8 \qquad (30)$$
$$k_{11} = -k_4 + k_7 \qquad (31)$$
$$k_{12} = -k_1 - k_7 \qquad (32)$$
$$k_{13} = k_4 - k_5 - 2k_6 \qquad (33)$$

The solution matrix can now be written by inspection of equations (29) through (33).

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	k_{13}
	h'	$\Delta T'$	H'	$\overline{\rho}'$	V_s'	g_o'	c_p'	a'	$\overline{\beta}'$	W'	μ'	k'	ω'
π_1	1	0	0	0	0	0	0	0	0	1	0	-1	0
π_2	0	1	0	0	0	0	0	0	1	0	0	0	0
π_3	0	0	1	0	0	0	0	0	0	-1	0	0	0
π_4	0	0	0	1	0	0	0	0	0	2	-1	0	1
π_5	0	0	0	0	1	0	0	0	0	-1	0	0	-1
π_6	0	0	0	0	0	1	0	0	0	-1	0	0	-2
π_7	0	0	0	0	0	0	1	0	0	0	1	-1	0
π_8	0	0	0	0	0	0	0	1	0	-1	0	0	0

The rank of the solution matrix is equal to the number of rows; thus the rows are linearly independent which insures that the dimensionless parameters π_1 through π_8 are also linearly independent.

The resulting dimensionless parameters are:

$$\pi_1 = \frac{h'W'}{k'} \qquad \pi_2 = \overline{\beta}'\Delta T' \qquad \pi_3 = \frac{H'}{W'}$$

$$\pi_4 = \frac{\overline{\rho}'W'^2\omega'}{\mu'} \quad \pi_5 = \frac{V_s'}{W'\omega'} \qquad \pi_6 = \frac{g_o'}{W'\omega'^2}$$

$$\pi_7 = \frac{c_p'\mu'}{k'} \qquad \pi_8 = \frac{a'}{W'}$$

Recalling that g_o' and $\overline{\beta}'$ must appear in the dimensionless parameters as a product due to the form of the governing equations, we have:

$$\pi_9 = \pi_2 \ \pi_6 = \frac{\overline{\beta}'\Delta T' g_o'}{W'\omega'^2}$$ and the

number of dimensionless parameters is reduced to seven.

We can obtain the more familiar dimensionless parameters by combining some of the π's in the following manner:

$$\pi_{10} = \pi_9 \ \pi_4^2 \ \pi_7 = \frac{g'\overline{\beta}'\Delta T' W'^3 \overline{\rho}'^2 c_p'}{\mu' k'} \qquad \equiv \quad \text{Rayleigh number}$$

$$\pi_{11} = \pi_4 \ \pi_8 = \frac{\overline{\rho}'(\omega' a')W'}{\mu'} \qquad \equiv \quad \text{vibration Reynolds number}$$

and

$$\pi_{12} = \pi_8 \ \pi_5^{-1} = \frac{a'\omega'}{V_s'} \qquad \equiv \quad \text{form of Mach number.}$$

The seven pertinent dimensionless parameters forming a complete set are therefore:

$$\pi_1 = \frac{h'W'}{k'} \qquad \equiv \quad \text{Nusselt number}$$

$$\pi_3 = \frac{H'}{W'} \qquad \equiv \quad \text{aspect ratio}$$

$$\pi_7 = \frac{c'_p \mu'}{k'} \quad \equiv \quad \text{Prandtl number}$$

$$\pi_8 = \frac{a'}{W'} \quad \equiv \quad \text{amplitude ratio}$$

$$\pi_{10} = \frac{g' \overline{\beta}' \Delta T' W'^3 \overline{\rho}'^2 c'_p}{\mu' k'} \quad \equiv \quad \text{Rayleigh number}$$

$$\pi_{11} = \frac{\overline{\rho}' (\omega' a') W'}{\mu'} \quad \equiv \quad \text{vibration Reynolds number}$$

and

$$\pi_{12} = \frac{a' \omega'}{V'_s} \quad \equiv \quad \text{form of mach number.}$$

The Nusselt number may be expressed in terms of the other dimensionless parameters through use of a functional relation of the form

$$\pi_1 = \phi\ (\pi_3, \pi_7, \pi_8, \pi_{10}, \pi_{11}, \pi_{12}).$$

The experimental program will be designed so that π_3, π_7, π_8, π_{10}, π_{11}, and π_{12} may be varied and the function ϕ determined.

EXPERIMENTAL

Once the decision to study a rectangular geometry had been reached, the design criteria for the construction of an experimental test cell could then be established. The limiting factor governing test cell design was that the electrodynamic vibrations facility to be used could accommodate a maximum load of forty pounds if the desired acceleration level of 25g's was to be reached. Another factor was the necessity of maintaining a sufficient cell depth in order to assure one dimensionality. After a brief study it was decided to maintain a depth to width ratio of at least four with the conviction that this would minimize end effects and produce data which was essentially one dimensional. Other design criteria were established and a computer program written in order to select the most desirable range of test cell sizes. This was done and the first cell to be constructed used the dimensions shown below in Table 3.

Height - 16" ; Depth - 7"

Using 3 sets of sideplates

W'	H'/W'	D'/W'
1.7"	9.3	4.1
1.0"	15.8	7.0
0.5"	31.0	14.0

Table 3. Test Cell No. 1 - Dimensions.

The design included an electrically heated hot plate using eight individually controlled electric heaters. This plate was guarded by the use of fourteen individually controlled electric heaters. The cold plate was designed to be maintained as a heat sink with the use of either water or a refrigerant. In the latter case the entire cold plate would be incorporated into the evaporator circuit of a refrigeration system. Side plates were constructed from one-half inch plexiglass especially selected for its thermal and optical characteristics.

In the process of designing this cell there arose many questions concerning the effect of the cell base plate design on the induced transverse vibration level. In order to answer these questions a dummy cell was constructed of approximately the same size, mass, and mass distribution that would exist with the test cell. This cell was instrumented with accelerometers and put through an extensive series of tests to determine the magnitude of transverse vibrations as well as the effect of base plate design on these vibrations. The results of these tests are shown in Figures 5 and 6. Attention is directed to the marked decrease in the induced transverse vibration by modifying the base plate geometry.

With this and other background information the first test cell was constructed as shown in Figures 7 and 8. The completed test cell minus instrumentation is shown in Figures 9 and 10.

The instrumentation of the test cell was divided into three areas: thermal, vibration, and flow visualization. Since small vibration transducers could be attached at any point on the cell, the main instrumentation problem was that of thermal instrumentation. It was necessary to measure local heat flux and temperatures at many points in order to produce high integrity data. This was accomplished by providing eight primary heaters and fourteen guard heaters each individually controlled and instrumented so that power dissipation in each heater could be measured. In order to insure that sufficient information would be available concerning the thermal field an extensive matrix of thermocouples was installed. Twelve thermocouples were imbedded in the surface of the cold plate, sixteen imbedded in the surface of the hot plate, twenty-eight were used in association with the guard heaters, and seven were installed as traversing thermocouples. These thermocouples were suspended from the hot plate to the cold plate across the enclosure. A special mounting and sealing device was constructed so that these thermocouples could be moved longitudinally thus placing the thermocouple junction at any position across the width of the cell. The use of these seven traversing thermocouples will produce detailed data on temperature profiles within the cell. Some difficulty was encountered in constructing a butt welded thermocouple for use in this application. A special jig was built and used in the welding process in order to insure uniformity. Another jig was built upon which each thermocouple was tested to insure that it would withstand the vibration stresses which would be imposed upon it under test conditions. These thermocouples were also tested to insure that they will withstand the tension necessary to prevent large deflections at the center which would result in undesirable stirring of the fluid.

After an intensive search of the literature concerning flow visualization techniques, one method was selected for use in this research project. The one selected makes use of neutrally buoyant particles and slit illumination combined with photographic techniques. The scheme for accomplishing flow visualization and local velocity measurements has been decided upon but the collection of this data has been postponed until the second year.

The completely instrumented test cell is shown in Figure 11 mounted on the shaker. Initial operation of this cell was accomplished with water as the cooling medium for the cold plate. These operations consisted of the time consuming process of balancing heater inputs in order to insure isothermality of the surface of the hot plate and a no flux condition at the rear and edges of the hot plate. Once this was accomplished the first set of data was recorded.

It was the purpose of the first set of experiments to test the integrity of the apparatus in a non-vibratory mode. This was accomplished using a maximum hot plate temperature of 215^{o}F and a cold plate temperature of about 66^{o}F. The maximum deviation from an isothermal condition on the hot plate was 2^{o}F. This deviation occurred at a temperature of 215^{o}F and decreased significantly at lower temperatures. The maximum temperature variation over the surface of the cold plate was found to be 1.5^{o}F. Under these conditions power inputs were measured and temperatures recorded. After a radiation correction had been applied, the data was calculated and presented in the form of Rayleigh number versus Nusselt number variation.

The results of this first set of data are shown in Figures 12 and 13. The data shown in Figure 12 was submitted to a least squares curve which resulted in the equation

$$Nu = 0.233\ Ra^{0.248} \qquad (34)$$

Numbers were then computed using this equation and compared to experimental values and the results shown in Figure 13. The average per cent error and the sum of the errors squared are 6.84 and 2.12 respectively. These are based on the difference between the experimentally determined Nusselt number and the Nusselt number calculated using the above equation.

Preliminary comparison of this data to other information found in the literature indicates that the test apparatus is performing satisfactorily and yields high integrity data. It thus remains to begin a comprehensive testing program in the vibratory mode.

PROPERTIES STUDY

The molecular theory of liquids as proposed by several authors was studied in an effort to determine the effect, if any, of vibrations upon the transport properties of liquids. Among the theories investigated were those of Born and Green (188), Enskog (210) and Eyring (235). The theory of Born and Green could in principle be applied to the case in which an external force field is impressed upon a liquid. However, the experimental information available as to intermolecular potentials and collision cross-sections is not sufficient to allow even approximate solutions to the governing equations for this model.

The theory of dense gases and liquids as developed by Enskog is essentially a correction to the theory of dilute gases which takes into account the finite size of the molecules involved. The method yields results which are more accurate than one might expect from such an analysis. In order to utilize this theory in the present problem, it would be necessary to first obtain the effect of vibrations upon the transport properties of dilute gases.

Eyring's significant structure theory of liquids is essentially a hole-theory. According to this theory, a liquid contains a certain number of holes into which molecules of liquid may jump. The molecules which are making the jump into holes have gas-like properties, while the molecules remaining in essentially a fixed position have solid-like properties. The properties of the liquid may be determined from a weighted average of the properties of the solid and gaseous phases of the material. Although it would seem that vibratory motion impressed upon the liquid could affect the rate at which molecules tend to jump into holes, no reasonable method for analytically predicting such an effect could be found.

The difficulty of analytically predicting the effects of vibration upon the transport properties of liquids (or gases) lead to the conclusion that experimental investigation should be undertaken in order to determine whether or not the transport properties of liquids are affected by vibrations. In view of the fact that the effect of vibrations upon transport properties could not be determined analytically, the energy input to ideal gases as a result of vibrations was studied from the molecular theory standpoint.

The development which follows is based upon the kinetic theory of a hard-sphere ideal gas as formulated by Lee, Sears and Turcotte (217). The problem is formulated as follows. Consider a rigid container as shown in Figure 14 which encloses an ideal gas initially at some temperature (T_o) and pressure (P_o). The container is subjected to vibratory motion normal to one wall. The time rate of energy input due to the vibrations is determined as a function of the parameters of vibratory motion, gas properties, and container dimensions.

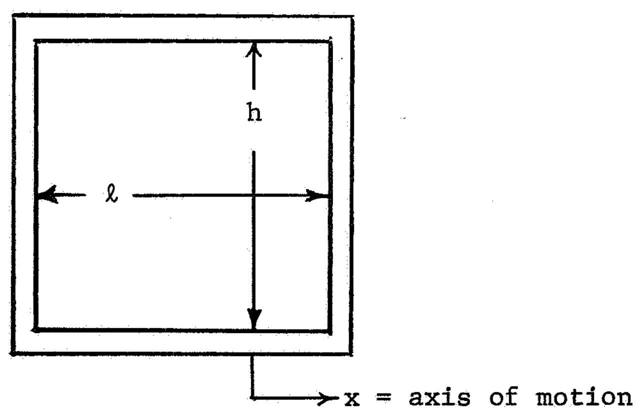

Figure 14. Rigid Container.

Consider a particle approaching a moving wall as shown in Figure 15.

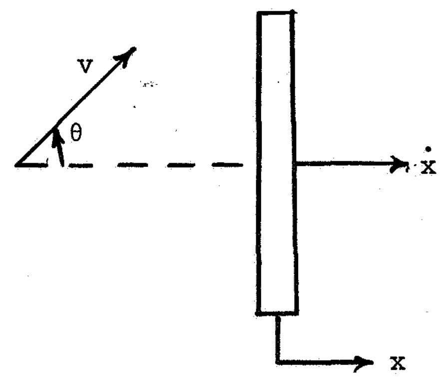

Figure 15. Nomenclature for Particle Approaching Moving Wall.

The relative velocity (x-component) is:

$$v \cos \theta - \dot{x}$$

If $v \cos \theta > \dot{x}$, then a collision will result. If one assumes a perfectly elastic collision, the speed (x-component) of the particle after collision will be

$$v \cos \theta - 2\dot{x}$$

The change of particle kinetic energy as a result of the collision will be:

$$\begin{aligned} \Delta KE &= \frac{m}{2} \left[(v \cos \theta - 2\dot{x})^2 - (v \cos \theta)^2 \right] \\ &= \frac{m}{2} \left[-4 v \cos \theta \, \dot{x} + 4\dot{x}^2 \right] \\ &= 2m \left[\dot{x}^2 - v \, \dot{x} \cos \theta \right] \end{aligned}$$

This is the change of internal energy of the gas due to the collision of a single particle with the wall. The rate of change of internal energy may be found by multiplying by the number of collisions per second. The number of collisions per unit area per unit time due to particles approaching the wall at angle between θ and $\theta + d\theta$ and traveling at speed between v and v + dv is given by

$$d^2 n_{v,\theta} = \frac{1}{2}(v \cos \theta - \dot{x}) dn_v \sin \theta \, d\theta$$

Where $d^2 n_{v,\theta}$ is the number of collisions per unit area per unit time.

$$\frac{\Delta KE}{\text{Collision}} \times \frac{\text{Collisions}}{\text{Area} - \text{time} - \theta - v} = \frac{\Delta KE}{A - t - \theta - v} = d^2\dot{E}$$

Where the $d^2\dot{E}$ notation is adopted for convenience. The d^2 implies differentials due to the ranges of v and θ being considered. $\dot{E}$ is the time rate of change of internal energy per unit area.

$$d^2\dot{E} = \frac{\Delta KE}{\text{Collision}} \cdot d^2 n_{v,\theta} = m \left[-v^2 \dot{x} \cos^2 \theta + 2v\dot{x}^2 \cos \theta + \dot{x}^3 \right] dn_v \sin \theta d\theta$$

To get the work due to molecules striking the wall from all possible angles, integrate from $\theta = 0$ to $\theta = 90^\circ$.

$$d\dot{E} = \int_0^{90^\circ} d^2\dot{E} = \int_0^{90^\circ} -m dn_v \left[v^2\dot{x} \cos^2\theta - 2v\dot{x}^2 \cos \theta + \dot{x}^3 \right] \sin \theta d\theta$$

$$d\dot{E} = -m dn_v \left[\frac{v^2\dot{x}}{3} - v\dot{x}^2 + \dot{x}^3 \right]$$

where $d\dot{E}$ is the change of internal energy per unit area per unit time due only to those molecules having speed between v and v + dv.

This equation is written for the case where the wall is moving away from the center of mass of the gas. The equation gives the net change of internal energy at the receding wall per unit area per unit time by those gas particles which strike the wall with velocities between v and v + dv. To determine the change of internal energy per unit area per unit time, integration over all particle velocities from 0 to ∞ is required. Since we are dealing with a rigid container here, the opposite (approaching) wall will be considered before carrying out the integration over all velocities. The situation at the approaching wall is shown in Figure 16.

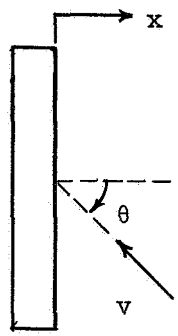

Figure 16. Conditions at Approaching Wall.

The x-component of speed relative between the particle and the wall before collision is

$$v \cos \theta + \dot{x}$$

and the x-component of speed after collision is

$$v \cos \theta + 2\dot{x}$$

The change of particle kinetic energy as a result of the collision is

$$\Delta KE = \frac{m}{2} \left[(v \cos \theta + 2\dot{x})^2 - (v \cos \theta)^2 \right]$$

$$= 2 \, m \left[v \, \dot{x} \cos \theta + \dot{x}^2 \right]$$

If collisions occur at same rate on both walls, the net ΔKE would be sum of the two ΔKE's. However, collisions will occur more rapidly at the wall where the relative velocity is v $\cos\theta + \dot{x}$. Therefore, find $d\dot{E}$ for the approaching wall and then add. For the approaching wall

$$d\dot{E} = \int_0^{90^o} mdn_v \left[v\dot{x}\cos\theta + \dot{x}^2 \right]\left[v\cos\theta + \dot{x} \right] \sin\theta d\theta$$

$$= mdn_v \left[\frac{v^2\dot{x}}{3} + v\dot{x}^2 + \dot{x}^3 \right]$$

The net (change of internal energy) is the sum of the changes at the two walls.

$$d\dot{E}_{net} = 2\ m\ v\ \dot{x}^2\ dn_v$$

This is the net change of internal energy of the gas per unit area per unit time due to molecules with velocity between v and v + dv. Integrate over all velocities and get

$$\dot{E}_{net} = \int_0^{\infty} 2mv\dot{x}^2 dn_v = 2m\dot{x}^2\bar{v}\frac{N}{V}$$

where

N = number of molecules

V = total volume of container

$\bar{v}$ = average speed of molecules

Now, it can be shown that (see reference (217))

$$\bar{v} = \text{average velocity} = \sqrt{\frac{2.55k}{m}T}$$

where T = absolute temperature

k = Boltzman's constant

m = mass of molecule.

Let

$$K_1 = \sqrt{\frac{2.55k}{m}}$$

Then

$$\dot{E}_{net} = \frac{2m\dot{x}^2 K_1 \sqrt{T}\; N}{V}$$

Since

$$mN = M = \text{total mass}$$

and

$$V = A\ell$$

$$\dot{E}_{net} = \frac{2M\dot{x}^2 K_1 \sqrt{T}}{A\ell}$$

which is the time rate of change of internal energy per unit area of the enclosure walls. The total time rate of change of internal energy is given by

$$\frac{dE}{dt} = \frac{2M\dot{x}^2 K_1 \sqrt{T}}{\ell}$$

For the case in which

$$\dot{x} = a'\omega' \sin a't$$

$$\frac{dE}{dt} = \frac{2Ma'^2\omega'^2 \sin^2 \omega't\; K_1 \sqrt{T}}{\ell}$$

Letting the acceleration in g's be given by

$$\acute{z} = \frac{a'\omega'^2}{32.2}$$

and substituting yields

$$\frac{dE}{dt} = \frac{2M\,(32.2)^2\, z^2 \sin^2 \omega't\; K_1 \sqrt{T}}{\ell\omega'^2}$$

The time rate of change of internal energy of the vibrating enclosure is proportional to the square of the acceleration level and inversely proportional to the square of the frequency.

In order to obtain some feel for the magnitude of this energy input, consider the case in which oxygen is enclosed within a rectangular container under the following conditions

$$\ell = 1 \text{ foot}$$

$$T = 600^{\circ}R$$

$$p = 1 \text{ atm}$$

$$\ddot{z} = 20 \text{ g's}$$

$$\omega' = 100 \text{ rad/sec.}$$

$$A = 1 \text{ ft}^2$$

For oxygen

$$K_1 = 62.9 \ \frac{\text{ft}}{\text{sec} - {}^{\circ}R^{\frac{1}{2}}}$$

Under these conditions

$$\frac{dE}{dt} = 0.339 \sin^2 \omega t \ \frac{\text{BTU}}{\text{sec}}$$

Taking the average rate of energy input over a cycle gives

$$\frac{d\bar{E}}{dt} = \frac{0.339}{2} = 0.170 \ \frac{\text{BTU}}{\text{sec}} \quad .$$

This value appears to be large enough to be significant in some cases. However, it can be noted that the frequency chosen is rather low - approximately 16 cycles per second. The amplitude of vibration is about 0.72 inches which is greater than one would expect to encounter in most applications.

Since there is a continuous energy input during vibrations of an ideal gas enclosed within a rectangular container, it is of some interest to investigate the maximum temperatures that would occur in this type situation. It is obvious that if a perfectly insulated container were used, the temperature would rise indefinitely. However, perfect insulators do not exist; a steady-state condition will be attained.

In order to determine the situation at steady-state, the rate of energy input due to vibrations is set equal to the rate of heat transfer out. Figure 17 indicates the nomenclature for this situation.

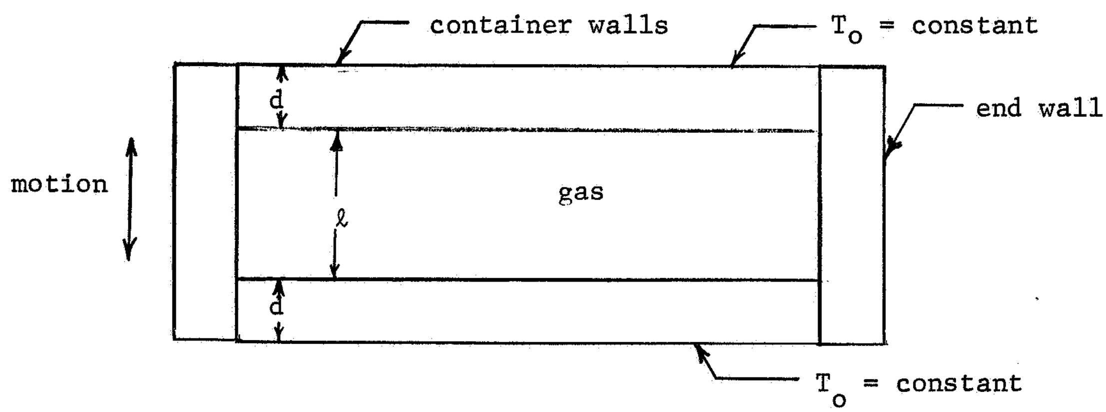

Figure 17. Nomenclature for Enclosed Vibrating Gas.

It is assumed that heat losses through the end walls of the container may be neglected. The temperature of the outside walls of the container is assumed to remain constant at T_o. Equating the averaged time rate of energy input due to vibrations to the rate of energy loss due to conduction gives

$$\frac{Ma'^2\omega'^2K_1\sqrt{T}}{\ell} = 2\,k_wA_w\,\frac{(T - T_o)}{d}$$

where

A_w = conducting wall area

k_w = conductivity of walls

Dividing by A_w and noting that $\frac{M}{V}$ equals density gives

$$\rho a'^2 \omega'^2 K_1 \sqrt{T} = \frac{2k_w(T - T_o)}{d}$$

Elimination of amplitude (a') in favor of acceleration level ($\ddot{z}$) and rearranging yields

$$T - \frac{(32.2)^2 \ddot{z}^2 \rho K_1 d}{2k_w \omega^2} \sqrt{T} + T_o = 0$$

Consider the case in which the gas is oxygen, and

d = 1/2 inch

$\ddot{z}$ = 100 g's

ω' = 2π x 100 rad/sec.

k_w = .09 ft - lb_f/sec. - ^oF - ft

T_o = 540^oR

p_o = 14.7 psia

Then

K_1 = 62.9 ft/sec - $^oR^{1/2}$

$$\rho = \frac{p_o}{RT_o} = .081 \; lb_m/ft^3$$

Solving for the steady-state temperature yields

$T = 564^oR$

Thus, a maximum temperature rise of 24^oF could be expected for this particular case.

CONCLUSIONS AND RECOMMENDATIONS

The work done during the past year on this research contract has established a firm basis from which the research can now be directed to systems utilizing cryogens as well as to systems of different geometrical configurations. A brief outline of the general results to date follows.

An experimental system utilizing air or water as the test fluid, was constructed for a rectangular cross-section enclosure with directly opposing isothermal vertical walls, and heat transfer data in a non-vibratory mode was obtained for one aspect ratio and various values of the Rayleigh number. The basic apparatus design proved to be satisfactory since heat transfer results measured under stationary conditions compared favorably with published results for similar geometrical configurations. A mathematical model for the rectangular configuration was proposed and its solution by finite difference techniques initiated and partial results obtained. The results of the analytical work were compared with those obtained experimentally and efforts at obtaining a semi-empirical equation for overall heat transfer rates were started. Several mathematical models were proposed for the thermal conductivity of gases and liquids in attempts to predict the effects of forced vibrations on this property; the proposed models proved to be mathematically intractable. However, a study of these models indicated the need for experimental work in this area. An analytical prediction of the effect of vibrations upon the bulk temperature of an ideal gas thermally insulated from its surroundings was obtained.

The general results outlined above and presented in detail in the previous section will serve as a basis for our next year's work, which will be concerned with:

a) the completion of the numerical work on the mathematical model for the rectangular enclosure,

b) the establishment of the vibratory conditions under which the assumption of bulk motion is valid,

c) revise the mathematical model or develop a new one to allow deviations from the bulk motion assumption,

d) the completion of the experimental work with water as the confined fluid,

e) the derivation of a semi-empirical equation for the heat transfer rates for water as the confined fluid (by combining the experimental and analytical results),

f) the experimental determination of the heat transfer rates and temperature fields for a cryogen (LN_2) as the enclosed fluid (rectangular enclosure), and

g) the experimental determination of the effect of vibrations upon the thermal conductivity.

In the following paragraphs the objectives itemized above will be described in more detail, and the anticipated methods of attack will be outlined.

ANALYTICAL

Numerical solutions of the mathematical model proposed during the past year for the rectangular enclosure will be sought for an extended range of the pertinent parameters over which the model is anticipated to be valid. These additional results will be combined with the experimental results of the past year in an attempt to obtain a semi-empirical equation which will give overall heat transfer rates for various vibratory conditions. This equation would be valid only over the range of vibration variables, such as frequency and acceleration, where the assumption of "bulk-liquid" motion has been experimentally verified to be valid. Velocity and temperature fields obtained by numerical procedures will be compared with those obtained experimentally. Comparisons will also be made with the published results for the non-vibratory case which should show graphically the effects of the vibrations upon the temperature and velocity fields.

Once the range of validity of the "bulk-liquid" motion assumption is determined, a means to extend the range of the solution will be sought. This may be accomplished by modification of the programming technique or the use of a different mathematical model.

The objective will be an equation with an analytical base, containing validation and modification from the experimental data, which will permit design engineers to conveniently predict thermal and velocity fields of fluid system under vibratory stress.

EXPERIMENTAL

It is recommended that an experimental investigation of the effect of vibration on heat transfer to liquid nitrogen be conducted. This is to be accomplished for a series of nitrogen-filled rectangular containers. Four of the container walls will be insulated and two opposing walls will be held at uniform, but different, temperatures. While the collection of data for liquid nitrogen will be the primary effort it will be preceded by completion of testing using water as a test fluid. These two efforts will phase together nicely since continued testing with water can be carried on at the same time that the liquid nitrogen system is being designed.

The majority of the coming year's experimental effort will be spent searching for resonance conditions and providing checks for any predictions which the analytical investigation may evolve. Subsequent to these tests a series of data collection runs will be carried out to obtain thermal, photographic, and vibration information over the available ranges of the variables involved. It is this latter part of testing which will occupy the initial part of the second year's effort.

The investigations using liquid nitrogen as the contained fluid will necessitate the use of a heat sink at a temperature lower than that of LN_2. This will be accomplished by placing the test cell in a larger chamber which will be partially filled with LN_2. If the pressure in the chamber is reduced to 2 psia, the corresponding temperature will be 115^oR. Maintenance of atmospheric, or greater, pressure in the cell will insure that a temperature of 138^oR, or greater, can be reached before boiling will occur. Such an arrangement will yield an allowable temperature difference across the cell of 23^oR or greater, depending on cell pressurization.

The specific experimental efforts that should be exerted during the year are itemized as follows:

(1) Collection of data over the full range of variables using the present water-filled test cell. (aspect ratios: 9.3, 15.8, 31.0)

(2) Design and construction of a liquid nitrogen test unit to provide a range of aspect ratios similar to

those of the water cells. Collection of data over the full range using this cell.

The experimental variables to be recorded are:

(1) Physical dimensions of each test unit to determine aspect ratios.

(2) Frequency, amplitude, acceleration level, and mode of imposed vibration.

(3) Photographic records of typical and atypical flow patterns.

(4) Sufficient data (temperatures, power input, etc.) to determine the Rayleigh number for each configuration and test point.

(5) Temperature distributions within the fluid under both static and vibratory conditions.

The range and density of these variables to be recorded will be:

(1) Two test units (1 - H_2O, 1 - LN_2) with three aspect ratios achievable for each unit.

(2) Zero to 4 kc/sec range on vibration frequency. Range or amplitude will be from zero to 1 inch and on acceleration from zero to 25g's. A sufficient density of points will be recorded to satisfactorily cover the range.

(3) Sufficient photographs will be made of the water-filled cell to provide a thorough description of flow patterns. No flow visualization will be attempted for

the LN_2-filled cells unless other data indicates the existance of some flow phenomena not found in the water-filled cells.

(4) Rayleigh number ranges from 10^4 to 10^9 will be recorded with approximately fifteen points per decade. This will be accomplished by suitable variation in wall temperature, heat flux, and aspect ratio.

(5) The seven thermocouple scheme for measuring fluid temperature distributions will be used for both cells. This produces the temperature profiles at seven vertical stations in the fluid.

(6) Any conditions of resonance or atypical flow will be closely monitored with all the experimental variables and detailed data will be recorded in these instances.

PROPERTIES STUDY

The analytical determination of the effect of vibrations upon transport properties of fluids is a rather formidable task. Any analytical results which have been obtained in the past or which may be obtained in the foreseeable future must be based upon molecular models which only approximate the actual physical situation. It is therefore essential that reliable information regarding vibration effects on transport properties be based upon experiment.

The transport properties which could be influenced by vibrations include thermal conductivity and viscosity. Of the two, the thermal conductivity appears to offer the greatest prospects for accurate measurement under vibratory conditions. Furthermore, this property appears at the moment more likely to be affected by vibrations than viscosity, because it has been demonstrated that the internal modes of motion of polyatomic molecules have appreciable effect upon the thermal conductivity but have little effect upon the viscosity. It would seem that if vibrations are to have any effect at all upon properties, the effect would be due to changes in relaxation time; i. e., effects would be due to the time required to transfer energy from translational to internal modes.

Therefore, it is recommended that the following investigations be undertaken during the course of this project:

(1) Obtain numerical results for the rectangular cross-section enclosure problem for several values of the pertinent dimensionless parameters and compare these results to those obtained in the experimental program;

(2) Obtain complete experimental data from both the water and the LN_2 test cells;

(3) Establish the vibratory conditions under which the assumption of bulk-liquid motion yields acceptable analytical predictions of heat transfer rates;

(4) Seek an alternate model which will relax the bulk-liquid motion assumption;

(5) Measure the thermal conductivity of selected gases and liquids under vibratory conditions and over temperatures ranging from ambient to cryogenic.

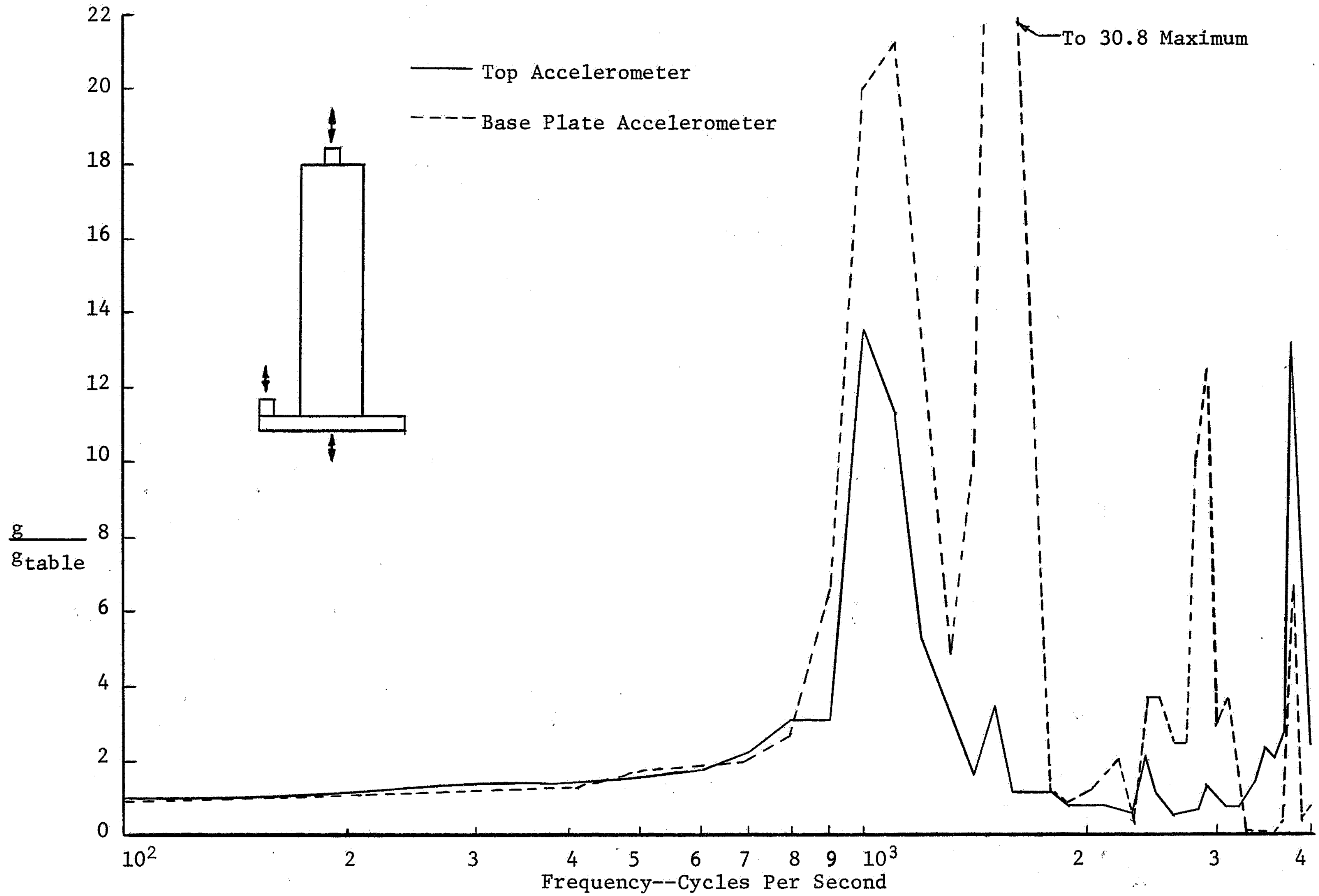

Figure 5. Relative Longitudinal Acceleration of the Dummy Cell.

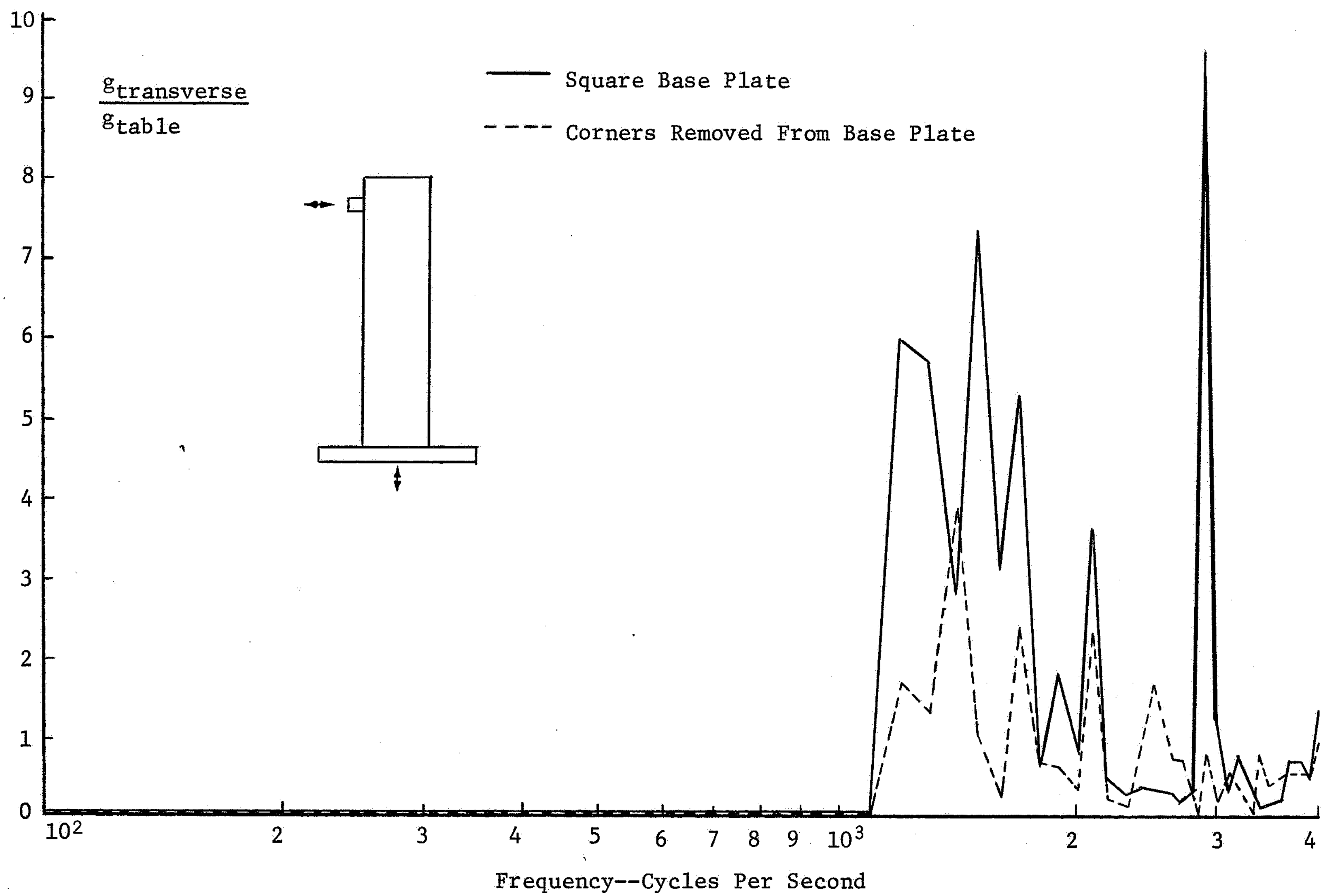

Figure 6. Relative Transverse Acceleration of Dummy Cell.

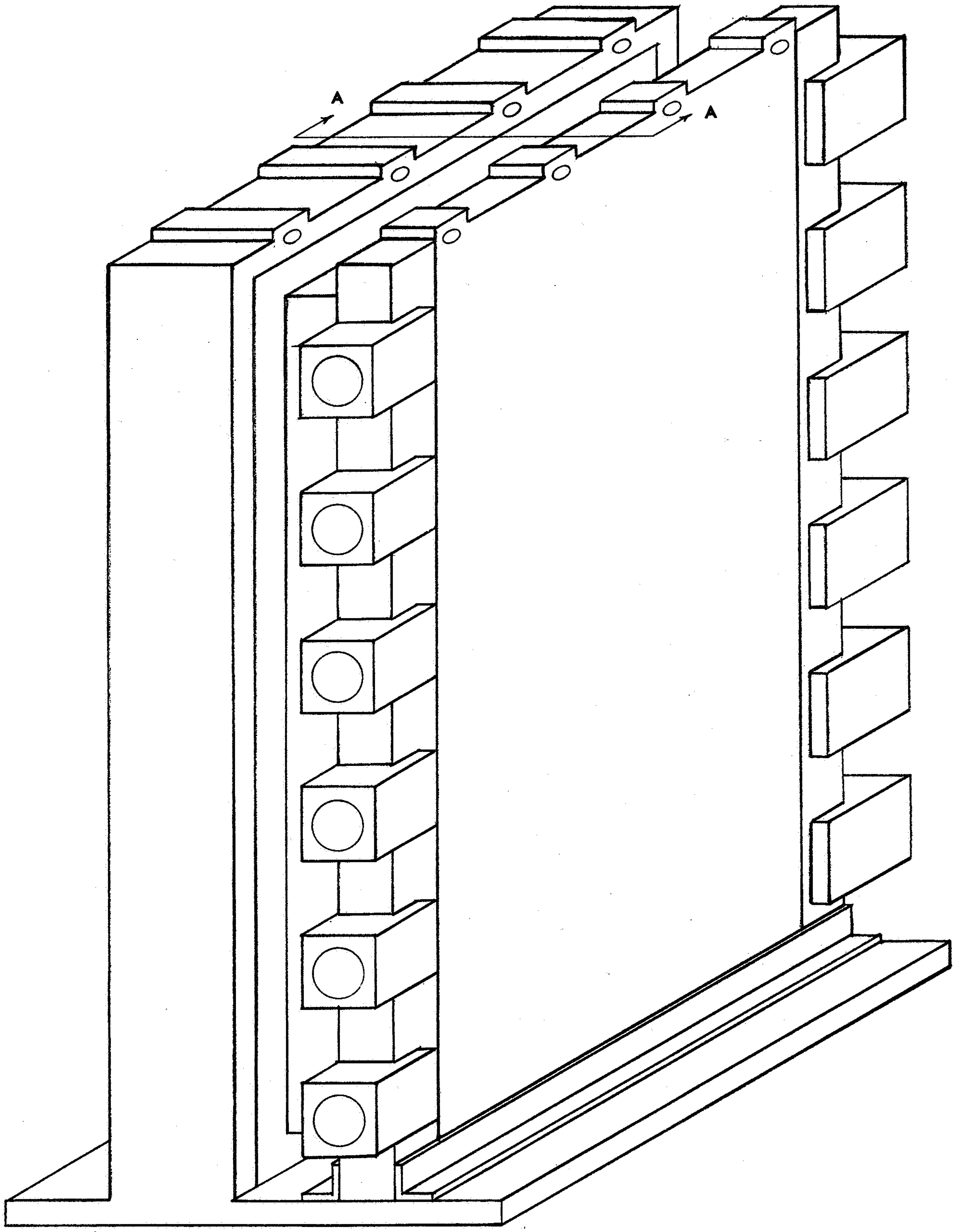

Figure 7. Construction Drawing of Complete Test Cell.

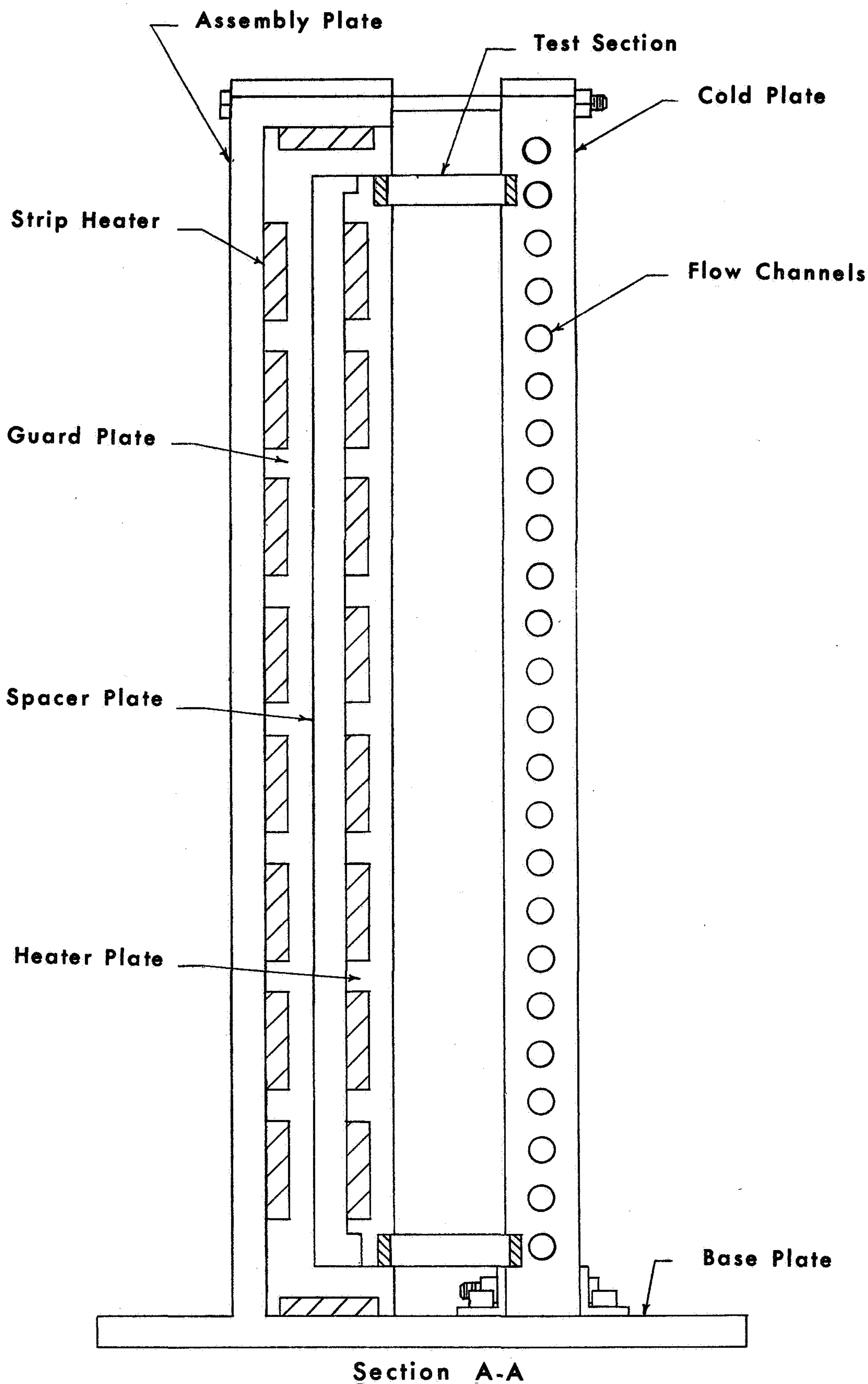

Figure 8. Cross Section of Test Cell.

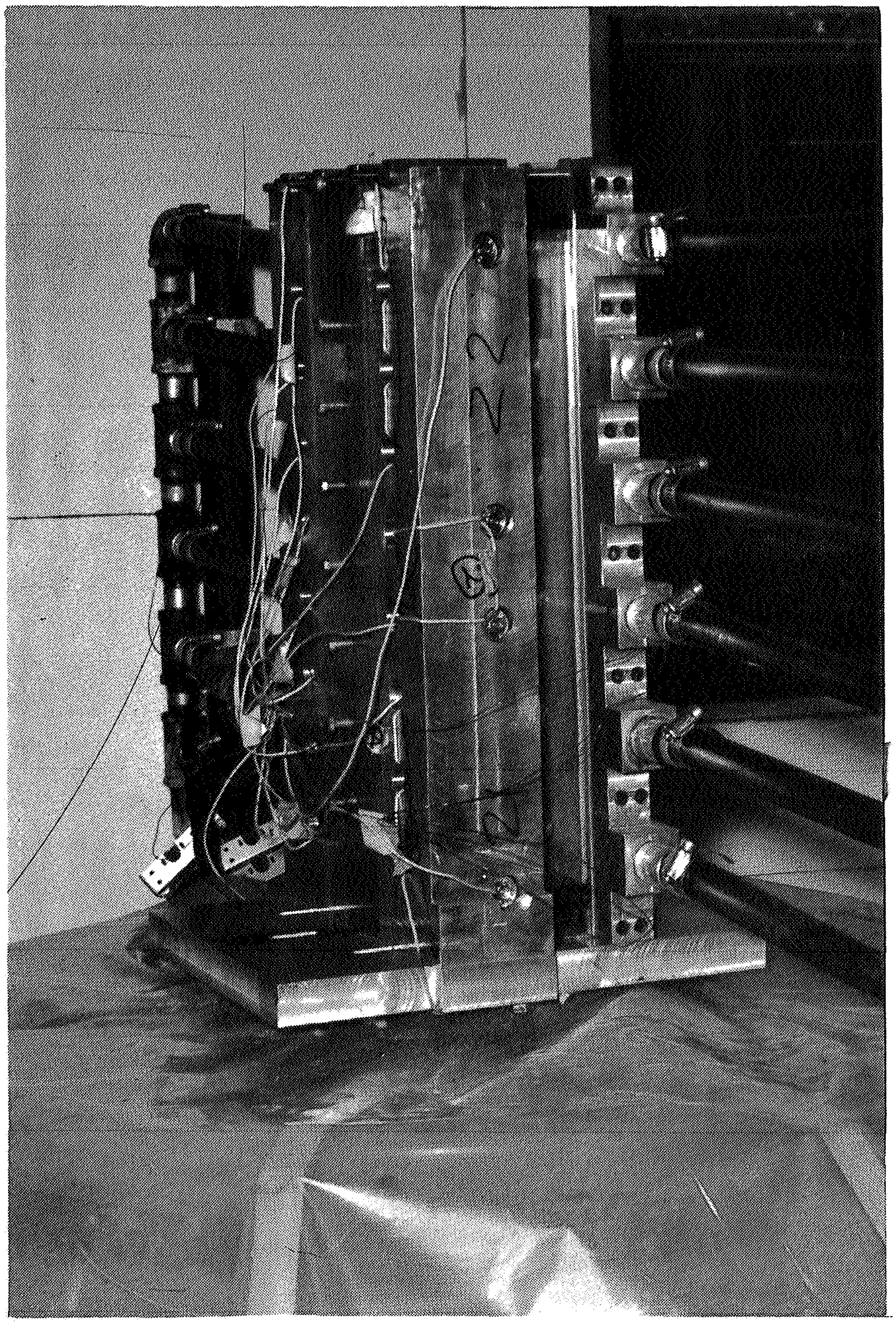

Figure 9. Assembled Test Cell #1 Aspect Ratio = 9.3.

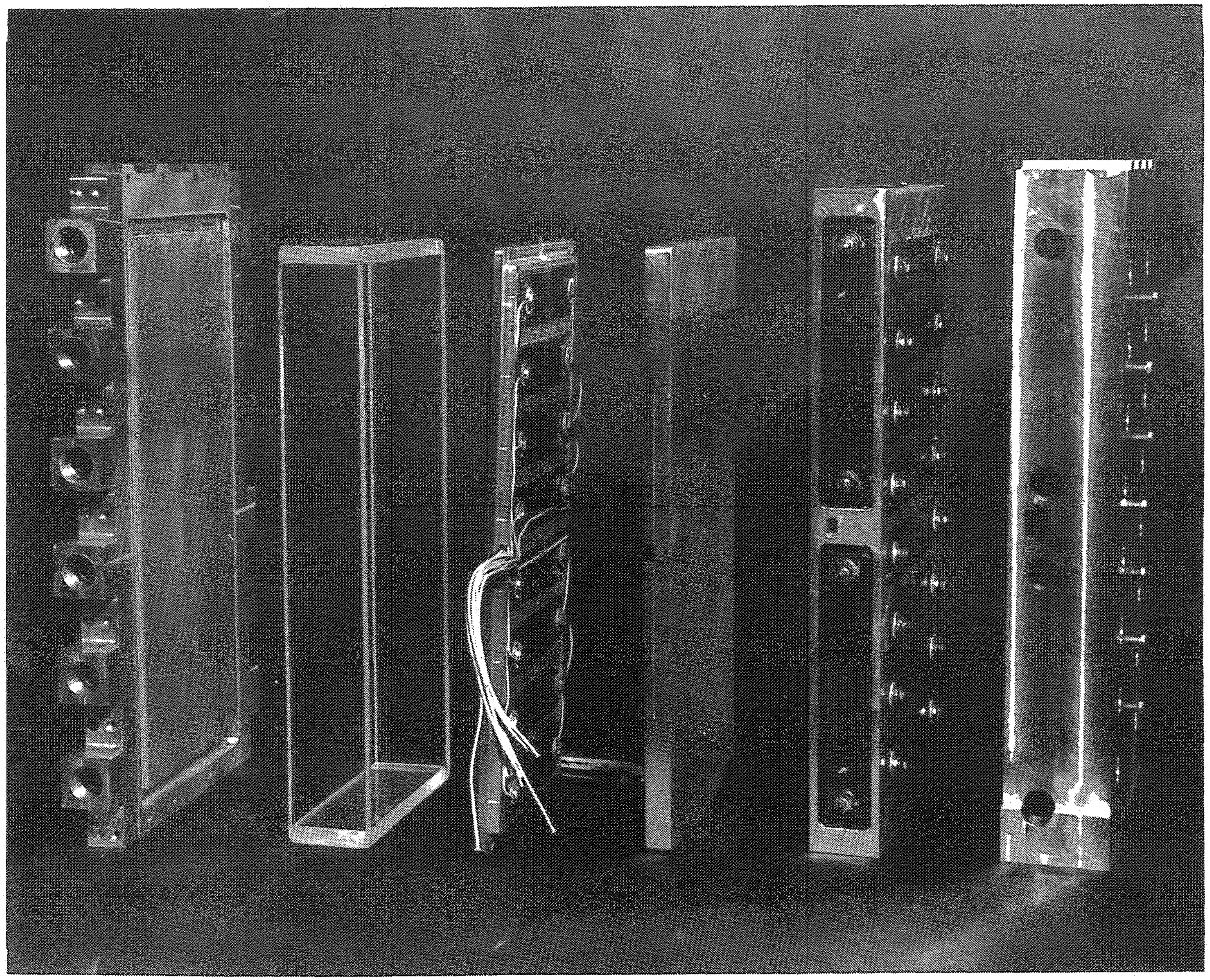

Figure 10. Exploded View of Test Cell #1.

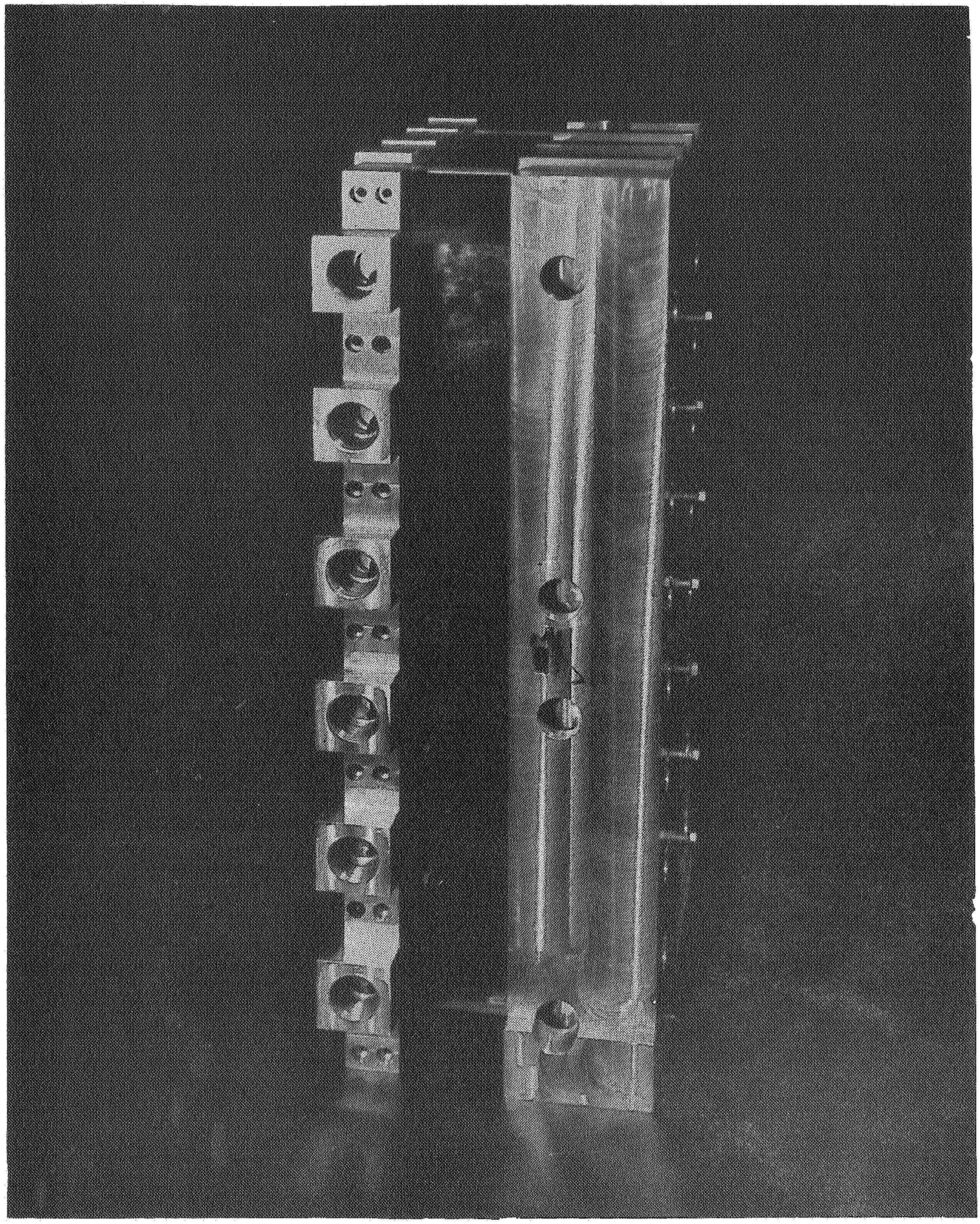

Figure 11. Completed Test Cell.

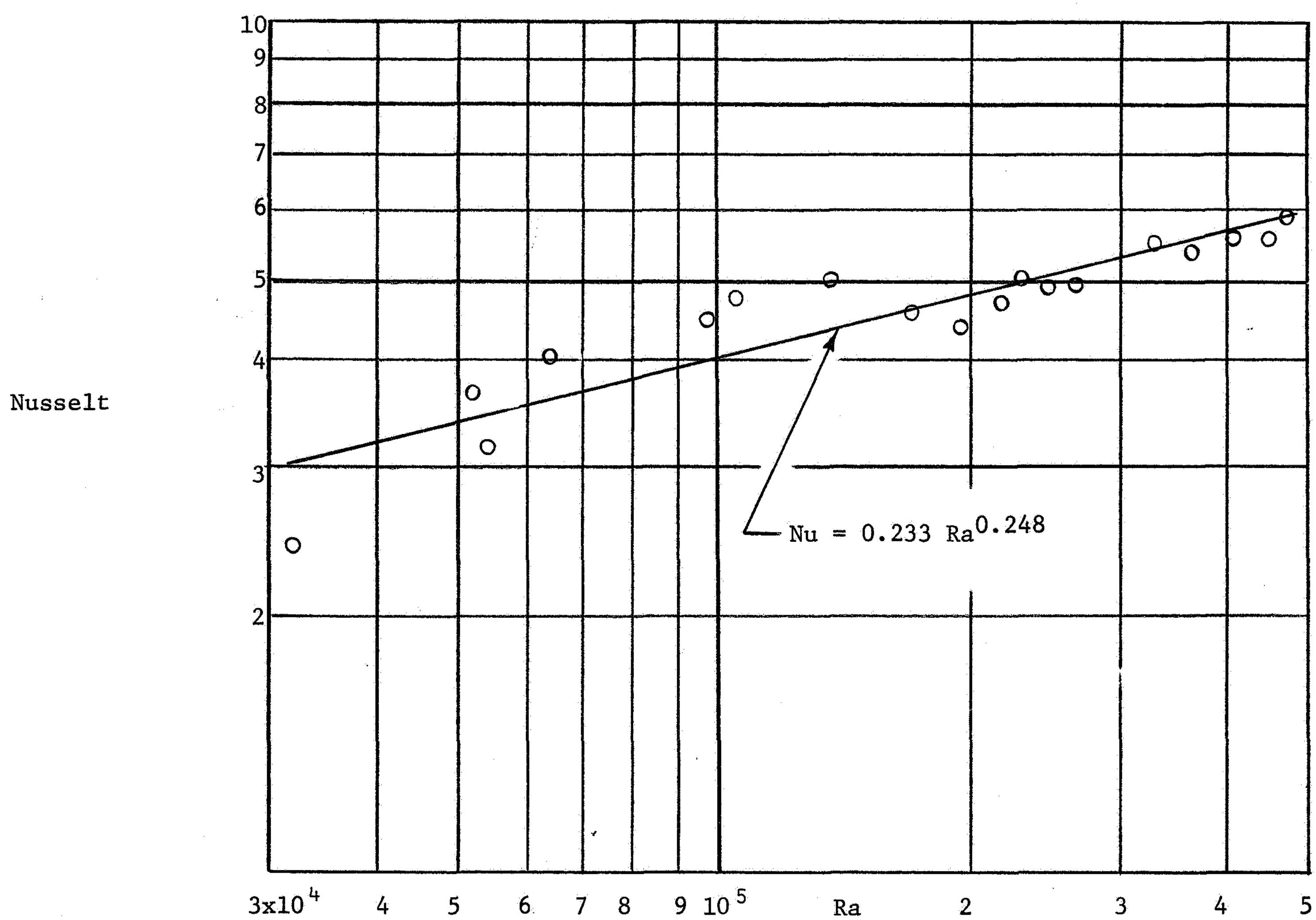

Figure 12. Cell Heat Transfer - Fluid; Air.

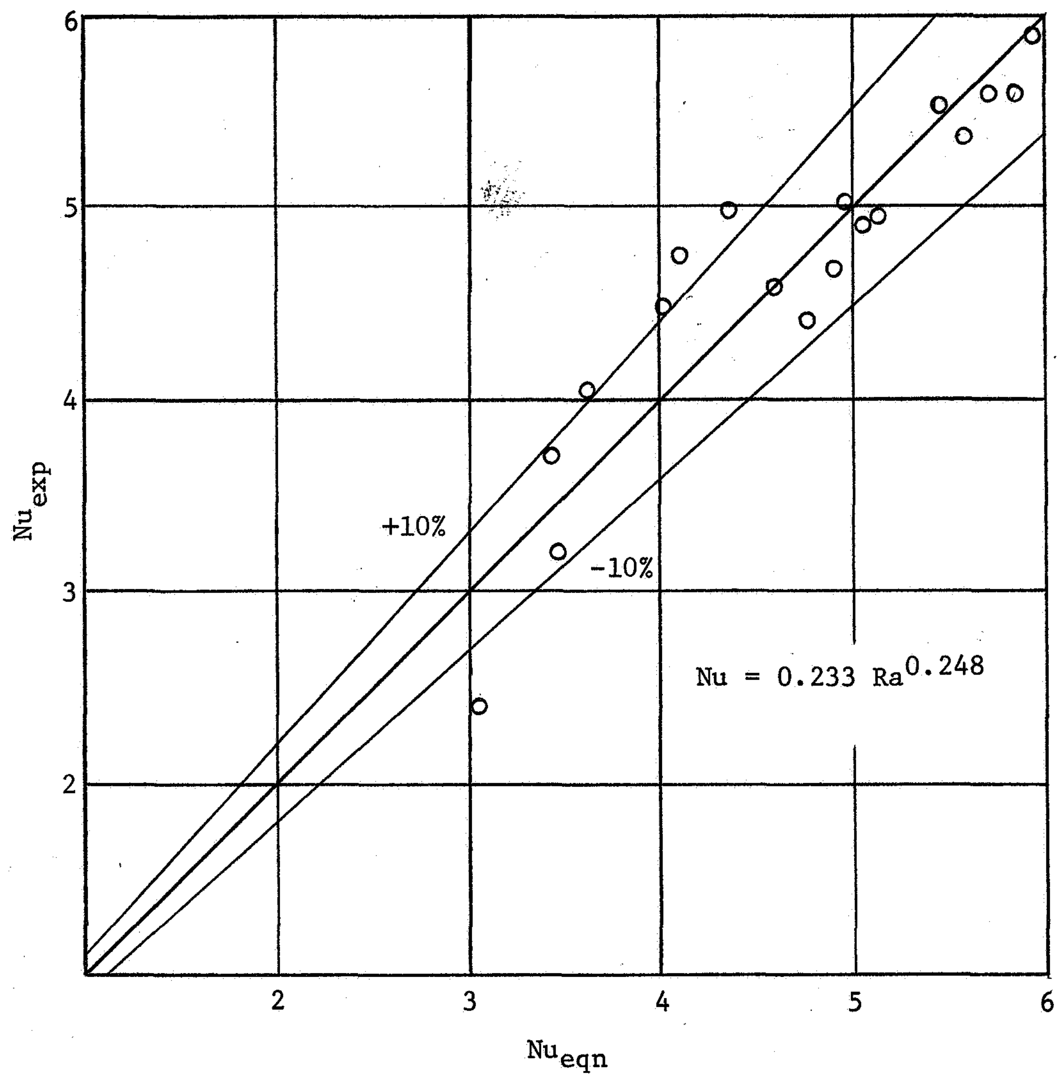

Figure 13. Experimental Nusselt Numbers Compared to Predictive Equation.

Appendix I

Finite Difference Formulation of the Natural Convection Problem

The system of partial differential equations governing convective heat transfer in a rectangular enclosure subjected to vibratory motion is given below along with suitable initial and boundary conditions. For the following discussion the vibratory motion will be considered longitudinal or parallel to the gravity vector.

The energy equation is:

$$Pr \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \nabla^2 T \tag{1a}$$

$$\text{where} \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \tag{2a}$$

$$v = -\frac{\partial \psi}{\partial x} . \tag{3a}$$

The defining equation for vorticity is:

$$\nabla^2 \psi = -\zeta . \tag{4a}$$

The combined momentum or vorticity equation is:

$$\frac{\partial \zeta}{\partial t} + \frac{u}{Pr} \frac{\partial \zeta}{\partial x} + \frac{v}{Pr} \frac{\partial \zeta}{\partial y} = \nabla^2 \zeta + Ra \frac{\partial T}{\partial x} \left(1 - \frac{g'}{g_0'} \sin \frac{\omega' W'^2}{\nu'} t \right) . \tag{5a}$$

The initial conditions are:

$$\psi(x, y, 0) = 0$$
$$u(x, y, 0) = 0$$
$$v(x, y, 0) = 0$$
$$\zeta(x, y, 0) = 0 \quad \text{and}$$

$$T(x,y,0) = 0 \quad \text{(uniform initial temperature).}$$

The appropriate boundary conditions are:

$$\Psi(0,y,t) = \Psi(1,y,t) = 0$$
$$\Psi(x,0,t) = \Psi(x,\tfrac{H'}{W'},t) = 0$$
$$u(x,0,t) = u(x,\tfrac{H'}{W'},t) = 0$$
$$v(0,y,t) = v(1,y,t) = 0$$
$$T(0,y,t) = 1$$
$$T(1,y,t) = 0 \qquad \text{and}$$
$$\frac{\partial T(x,0,t)}{\partial y} = \frac{\partial T(x,\frac{H'}{W'},t)}{\partial y} = 0 .$$

The system consists of the five equations (1a) through (5a) containing the five variables T, u, v, Ψ, and ζ.

Considering the finite difference approximation for the energy equation during the first half time step, using central differences,

$$Pr\left(\frac{T^*_{i,j} - T_{i,j}}{\Delta t/2}\right) + u_{i,j}\left(\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x}\right) + v_{i,j}\left(\frac{T^*_{i,j+1} - T^*_{i,j-1}}{2\Delta y}\right) = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} + \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta y)^2} \tag{6a}$$

where x space derivatives have been expressed explicitly and y space derivatives have been expressed implicitly. The superscript * will be used hereinafter to indicate the value of a dependent variable after a time increment $\Delta t/2$. The velocity terms u and v have been held constant during the half time step at their "old" value. During the second half time step:

$$Pr\left(\frac{T'_{i,j} - T^*_{i,j}}{\Delta t/2}\right) + u_{i,j}\left(\frac{T'_{i+1,j} - T'_{i-1,j}}{2\Delta x}\right) + v_{i,j}\left(\frac{T^*_{i,j+1} - T^*_{i,j-1}}{2\Delta y}\right) =$$

$$\frac{T'_{i-1,j} - 2T'_{i,j} + T'_{i+1,j}}{(\Delta x)^2} + \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta y)^2} \qquad (7a)$$

where x space derivatives have been expressed implicitly and y space derivatives have been expressed explicitly, again using central differences. The superscript ' will be used to designate the value of a dependent variable after a second half time step $\Delta t/2$ or after a total time Δt. The variables u and v have again been held constant during the second half time step.

Solving equation (6a) for T* in terms of T:

$$T^*_{i,j-1}\left(-\frac{v_{i,j}}{2\Delta y} - \frac{1}{(\Delta y)^2}\right) + T^*_{i,j}\left(\frac{2Pr}{\Delta t} + \frac{2}{(\Delta y)^2}\right)$$

$$+ T^*_{i,j+1}\left(\frac{v_{i,j}}{2\Delta y} - \frac{1}{(\Delta y)^2}\right) = T_{i-1,j}\left(\frac{u_{i,j}}{2\Delta x} + \frac{1}{(\Delta x)^2}\right)$$

$$+ T_{i,j}\left(\frac{2Pr}{\Delta t} - \frac{2}{(\Delta x)^2}\right) + T_{i+1,j}\left(-\frac{u_{i,j}}{2\Delta x} + \frac{1}{(\Delta x)^2}\right) \qquad (8a)$$

which holds for all interior grid points such that

$$2 \leq i \leq m - 1$$

$$2 \leq j \leq n - 1 \quad .$$

Considering $j = 1$, or along the bottom insulated wall: $u_{i,1} = 0$ and

$$\left(\frac{\partial T^*}{\partial y}\right)_{i,1} = 0 \qquad \text{thus} \qquad Pr\,\frac{\partial T}{\partial t} = \nabla^2 T$$

Considering a Taylor's series expansion about $T^*_{i,1}$ we have

$$T^*_{i,2} = T^*_{i,1} + \Delta y\left(\frac{\partial T^*}{\partial y}\right)_{i,1} + \frac{(\Delta y)^2}{2}\left(\frac{\partial^2 T^*}{\partial y^2}\right) + \cdots$$

thus

$$\left(\frac{\partial^2 T^*}{\partial y^2}\right)_{i,1} = \frac{2\left(T^*_{i,2} - T^*_{i,1}\right)}{(\Delta y)^2} .$$

Rewriting equation (6a) for the special case j = 1

$$Pr\left(\frac{T^*_{i,1} - T_{i,1}}{\Delta t/2}\right) = \frac{T_{i-1,1} - 2\,T_{i,1} + T_{i+1,1}}{(\Delta x)^2}$$

$$+ \frac{2\left(T^*_{i,2} - T^*_{i,1}\right)}{(\Delta y)^2} .$$

Solving for T* in terms of T

$$T^*_{i,1}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta y)^2}\right) + T^*_{i,2}\left(-\frac{2}{(\Delta y)^2}\right) =$$

$$\frac{\left(T_{i-1,1} + T_{i+1,1}\right)}{(\Delta x)^2} + T_{i,1}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta x)^2}\right) \tag{9a}$$

which holds for grid points such that

$$2 \leqslant i \leqslant m - 1$$

$$j = 1 .$$

Similarly along the top insulated wall where j = n

$$T^*_{i,m-1}\left(-\frac{2}{(\Delta y)^2}\right) + T^*_{i,m}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta y)^2}\right) =$$

$$\frac{\left(T_{i-1,m} + T_{i+1,m}\right)}{(\Delta x)^2} + T_{i,m}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta x)^2}\right) \tag{10a}$$

which holds for grid points such that

$$2 \leqslant i \leqslant m - 1$$

$$j = n .$$

Thus equations (8a), (9a), and (10a) give the new values of temperature T*, after a time $\Delta t/2$, in terms of the old temperature T, Pr, u, v, Δx, Δy, and Δt. The matrix of this system has zeros everywhere except on the main diagonal and on the two diagonals parallel to and on either side of the main diagonal. Such a system is called a tridiagonal system and may be solved directly using a simple iterative method discussed by Wilkes (180) and Forsythe (64). Subsequent consideration of the energy and vorticity equations in this report will always lead to a tridiagonal system and it is for this very reason that the implicit alternating direction method is utilized.

For the second half time step, equation (7a) may be solved for T' in terms of T* giving

$$T'_{i-1,j}\left(-\frac{u_{i,j}}{2\Delta x}-\frac{1}{(\Delta x)^2}\right)+T'_{i,j}\left(\frac{2\,Pr}{\Delta t}+\frac{2}{(\Delta x)^2}\right)$$

$$+T'_{i+1,j}\left(\frac{u_{i,j}}{2\Delta x}-\frac{1}{(\Delta x)^2}\right)=T^*_{i,j-1}\left(\frac{v_{i,j}}{2\Delta y}+\frac{1}{(\Delta y)^2}\right)$$

$$+T^*_{i,j}\left(\frac{2\,Pr}{\Delta t}-\frac{2}{(\Delta y)^2}\right)+T^*_{i,j+1}\left(-\frac{v_{i,j}}{2\Delta y}+\frac{1}{(\Delta y)^2}\right) \qquad (11a)$$

which holds for grid points

$$3 \leqslant i \leqslant m-2$$

$$2 \leqslant j \leqslant n-1$$

during the second half time step. Special forms of equation (11a) will now be developed for the remaining points of the grid.

Considering the special case

$$i = 2 \quad , \quad j = 1$$

we have $\qquad u_{2,1} = 0$

$$\left(\frac{\partial T^*}{\partial y}\right)_{2,1} = 0$$ and considering

the Taylor's series expansion about $T^*_{2,1}$ we have

$$\left(\frac{\partial^2 T^*}{\partial y^2}\right)_{2,1} = \frac{2\,(T^*_{2,2} - T^*_{2,1})}{(\Delta y)^2} \quad .$$

The energy equation becomes, upon substitution of the above, and noting that $T'_{1,1} = T^*_{1,1} = T_{1,1}$ due to the isothermal condition

$$T'_{2,1}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) + T'_{3,1}\left(-\frac{1}{(\Delta x)^2}\right) =$$

$$T^*_{2,1}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{2,2}\left(\frac{2}{(\Delta y)^2}\right) + T_{1,1}\left(\frac{1}{(\Delta x)^2}\right) \qquad (12a)$$

which holds for the point $i = 2$, $j = 1$.

Using the same method, equations may be developed for the points

$$i = m - 1 \;,\; j = 1$$
$$i = 2 \qquad ,\; j = n$$
$$i = m - 1 \;,\; j = n \quad .$$

These equations may also be written by inspection of equation (12a) and varying subscripts. The equations are for

$$i = m - 1 \;,\; j = 1$$

$$T'_{m-2,1}\left(-\frac{1}{(\Delta x)^2}\right) + T'_{m-1,1}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) =$$

$$T^*_{m-1,1}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{m-1,2}\left(\frac{2}{(\Delta y)^2}\right) + T_{m,1}\left(\frac{1}{(\Delta x)^2}\right) ; \qquad (13a)$$

for i = 2 , j = n

$$T'_{2,m}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) + T'_{3,m}\left(-\frac{1}{(\Delta x)^2}\right) =$$

$$T^*_{2,m}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{2,m-1}\left(\frac{2}{(\Delta y)^2}\right) + T_{1,m}\left(\frac{1}{(\Delta x)^2}\right) ; \qquad (14a)$$

and for i = m - 1 , j = n

$$T'_{m-2,m}\left(-\frac{1}{(\Delta x)^2}\right) + T'_{m-1,m}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) =$$

$$T^*_{m-1,m}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{m-1,m-1}\left(\frac{2}{(\Delta y)^2}\right) + T_{m,m}\left(\frac{1}{(\Delta x)^2}\right) . \qquad (15a)$$

For intermediate points along j = 1 (i. e., $3 \leqslant i \leqslant m - 2$, j = 1) noting that $u_{i,1} = 0$,

$$\left(\frac{\partial T^*}{\partial y}\right)_{i,1} = 0 \quad \text{and} \quad \left(\frac{\partial^2 T^*}{\partial y^2}\right)_{i,1} = \frac{2\,(T^*_{i,2} - T^*_{i,1})}{(\Delta y)^2}$$

we have, upon substitution into the energy equation

$$T'_{i-1,1}\left(-\frac{1}{(\Delta x)^2}\right) + T'_{i,1}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) + T'_{i+1,1}\left(-\frac{1}{(\Delta x)^2}\right) =$$

$$T^*_{i,1}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{i,2}\left(\frac{2}{(\Delta y)^2}\right). \tag{16a}$$

A similar equation holds for intermediate grid points along j = n (i. e., $3 \leqslant i \leqslant m - 2$, $j = n$) thus

$$T'_{i-1,n}\left(-\frac{1}{(\Delta x)^2}\right) + T'_{i,n}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) + T'_{i+1,n}\left(-\frac{1}{(\Delta x)^2}\right) =$$

$$T^*_{i,n}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + T^*_{i,n-1}\left(\frac{2}{(\Delta y)^2}\right). \tag{17a}$$

Considering the special case of intermediate grid points along the column $i = 2$ (i. e., $i = 2$, $2 \leqslant j \leqslant n - 1$), noting that $T'_{1,j} = T^*_{1,j} = T_{1,j}$ due to the isothermal condition,

$$T'_{2,j}\left(\frac{2\,Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) + T'_{3,j}\left(\frac{u_{2,j}}{2\,\Delta x} - \frac{1}{(\Delta x)^2}\right) =$$

$$T^*_{2,j-1}\left(\frac{v_{2,j}}{2\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + T^*_{2,j}\left(\frac{2\,Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) +$$

$$T^*_{2,j+1}\left(-\frac{v_{2,j}}{2\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + T_{1,j}\left(\frac{u_{2,j}}{2\,\Delta x} + \frac{1}{(\Delta x)^2}\right). \tag{18a}$$

Similarly for intermediate points along the column $i = m - 1$ (i. e., $i = m - 1$, $2 \leqslant j \leqslant n - 1$)

$$T'_{m-2,j}\left(-\frac{u_{m-1,j}}{2\Delta x} - \frac{1}{(\Delta x)^2}\right) + T'_{m-1,j}\left(\frac{2Pr}{\Delta t} + \frac{2}{(\Delta x)^2}\right) =$$

$$T^*_{m-1,j-1}\left(\frac{v_{m-1,j}}{2\Delta y} + \frac{1}{(\Delta y)^2}\right) + T^*_{m-1,j}\left(\frac{2Pr}{\Delta t} - \frac{2}{(\Delta y)^2}\right) +$$

$$T^*_{m-1,j+1}\left(-\frac{v_{m-1,j}}{2\Delta y} + \frac{1}{(\Delta y)^2}\right) + T_{m,j}\left(-\frac{u_{m-1,j}}{2\Delta x} + \frac{1}{(\Delta x)^2}\right). \quad (19a)$$

Considering all special cases for the second half time step the resulting equations can be formed into three separate tridiagonal systems. The systems are formed by equations (12a), (13a), (16a)

by equations (14a), (15a), (17a)

and

by equations (11a), (18a), (19a).

Similar consideration will now be given to the vorticity equation and the resulting tridiagonal systems developed.

For the first half time step

$$\left(\frac{S^*_{i,j} - S_{i,j}}{\Delta t/2}\right) + \frac{u_{i,j}}{Pr}\left(\frac{S_{i+1,j} - S_{i-1,j}}{2\Delta x}\right) +$$

$$\frac{v_{i,j}}{Pr}\left(\frac{S^*_{i,j+1} - S^*_{i,j-1}}{2\Delta y}\right) = \left(\frac{S_{i-1,j} - 2S_{i,j} + S_{i+1,j}}{(\Delta x)^2}\right) +$$

$$\left(\frac{S^*_{i,j-1} - 2S^*_{i,j} + S^*_{i,j+1}}{(\Delta y)^2}\right) + Ra\ \ Vib\left(\frac{T'_{i+1,j} - T'_{i-1,j}}{2\Delta x}\right) \quad (20a)$$

where

$$Vib \equiv 1 - \frac{g'}{g_0'} \sin \frac{\omega' W'^2}{\nu'} t \qquad \text{and} \qquad t = \frac{\Delta t}{2} .$$

Note that new values of temperature have been used in equation (20a).

Solving (20a) for S^* in terms of S

$$S^*_{i,j-1}\left(-\frac{V_{i,j}}{2 Pr \Delta y} - \frac{1}{(\Delta y)^2}\right) + S^*_{i,j}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta y)^2}\right) +$$

$$S^*_{i,j+1}\left(\frac{V_{i,j}}{2 Pr \Delta y} - \frac{1}{(\Delta y)^2}\right) = S_{i-1,j}\left(\frac{U_{i,j}}{2 Pr \Delta x} + \frac{1}{(\Delta x)^2}\right) +$$

$$S_{i,j}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta x)^2}\right) + S_{i+1,j}\left(-\frac{U_{i,j}}{2 Pr \Delta x} + \frac{1}{(\Delta x)^2}\right) +$$

$$Ra\ Vib\left(\frac{T'_{i+1,j} - T'_{i-1,j}}{2 \Delta x}\right) \tag{21a}$$

which holds for grid points such that $2 \leqslant i \leqslant m-1$, $3 \leqslant j \leqslant n-2$.
When considering the case j = 2 the term $S^*_{i,1}$ is a vorticity along the boundary i,1 and since no new boundary vorticities are known at this stage old values of the boundary vorticities must be used for the term $S^*_{i,1}$.
Therefore for $2 \leqslant i \leqslant m-1$, $j = 2$ replacing $S^*_{i,1}$ by $S_{i,1}$

$$S^*_{i,2}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta y)^2}\right) + S^*_{i,3}\left(\frac{V_{i,2}}{2 Pr \Delta y} - \frac{1}{(\Delta y)^2}\right) =$$

$$S_{i-1,2}\left(\frac{U_{i,2}}{2 Pr \Delta x} + \frac{1}{(\Delta x)^2}\right) + S_{i,2}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta x)^2}\right) +$$

$$S_{i+1,2}\left(-\frac{U_{i,2}}{2 Pr \Delta x} + \frac{1}{(\Delta x)^2}\right) + S_{i,1}\left(\frac{V_{i,2}}{2 Pr \Delta y} + \frac{1}{(\Delta y)^2}\right) +$$

$$Ra \; Vib\left(\frac{T'_{i+1,2} - T'_{i-1,2}}{2\Delta x}\right) . \tag{22a}$$

Using s similar argument for the case $2 \leqslant i \leqslant m - 1$, $j = n - 1$

$$S^*_{i,m-2}\left(-\frac{v_{i,m-1}}{2\,Pr\,\Delta y} - \frac{1}{(\Delta y)^2}\right) + S^*_{i,m-1}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta y)^2}\right) =$$

$$S_{i-1,m-1}\left(\frac{u_{i,m-1}}{2\,Pr\,\Delta x} + \frac{1}{(\Delta x)^2}\right) + S_{i,m-1}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta x)^2}\right) +$$

$$S_{i+1,m-1}\left(-\frac{u_{i,m-1}}{2\,Pr\,\Delta x} + \frac{1}{(\Delta x)^2}\right) + S_{i,m}\left(-\frac{v_{i,m-1}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right)$$

$$+ Ra \; Vib\left(\frac{T'_{i+1,m-1} - T'_{i-1,m-1}}{2\Delta x}\right) . \tag{23a}$$

For the first half time step the tridiagonal system for the vorticity equation is formed using equations (21a), (22a), and (23a).

When considering the second half time step for the vorticity equation, the procedure is the same as for the first half time step with the following exceptions:

(1) x derivatives are expressed implicitly

(2) y derivatives are expressed explicitly

(3) the special cases are for

$$i = 2 \quad , \; 2 \leqslant j \leqslant n - 1$$
$$i = m - 1 \; , \; 2 \leqslant j \leqslant n - 1$$

(4) the t term in the expression

$$Vib = 1 - \frac{g'}{g'_o} \sin \frac{\omega' W'^2}{\nu'} t \qquad \text{is}$$

now Δt instead of $\Delta t/2$.

The resulting equations are for grid points $3 \leqslant i \leqslant m - 2$, $2 \leqslant j \leqslant n - 1$

$$S'_{i-1,j}\left(-\frac{u_{i,j}}{2\,Pr\,\Delta X} - \frac{1}{(\Delta X)^2}\right) + S'_{i,j}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta X)^2}\right) +$$

$$S'_{i+1,j}\left(\frac{u_{i,j}}{2\,Pr\,\Delta X} - \frac{1}{(\Delta X)^2}\right) = S^*_{i,j-1}\left(\frac{v_{i,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) +$$

$$S^*_{i,j}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta y)^2}\right) + S^*_{i,j+1}\left(-\frac{v_{i,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) +$$

$$Ra\ Vib\left(\frac{T'_{i+1,j} - T'_{i-1,j}}{2\,\Delta X}\right) ; \qquad (24a)$$

for grid points $i = 2$, $2 \leqslant j \leqslant n - 1$

$$S'_{2,j}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta X)^2}\right) + S'_{3,j}\left(\frac{u_{2,j}}{2\,Pr\,\Delta X} - \frac{1}{(\Delta X)^2}\right) =$$

$$S^*_{2,j-1}\left(\frac{v_{2,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + S^*_{2,j}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta y)^2}\right) +$$

$$S^*_{2,j+1}\left(-\frac{v_{2,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + S_{1,j}\left(\frac{u_{2,j}}{2\,Pr\,\Delta X} + \frac{1}{(\Delta X)^2}\right) +$$

$$Ra\ Vib\left(\frac{T'_{3,j} - T'_{1,j}}{2\,\Delta X}\right) ; \qquad (25a)$$

and for $i = m - 1$, $2 \leqslant j \leqslant n - 1$

$$S'_{m-2,j}\left(-\frac{u_{m-1,j}}{2\,Pr\,\Delta X} - \frac{1}{(\Delta X)^2}\right) + S'_{m-1,j}\left(\frac{2}{\Delta t} + \frac{2}{(\Delta X)^2}\right) =$$

$$S^*_{m-1,j-1}\left(\frac{v_{m-1,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + S^*_{m-1,j}\left(\frac{2}{\Delta t} - \frac{2}{(\Delta y)^2}\right) +$$

$$S^*_{m-1,j+1}\left(-\frac{v_{m-1,j}}{2\,Pr\,\Delta y} + \frac{1}{(\Delta y)^2}\right) + S_{m,j}\left(-\frac{u_{m-1,j}}{2\,Pr\,\Delta X} + \frac{1}{(\Delta X)^2}\right) +$$

$$Ra\ Vib\left(\frac{T_{m,j}-T'_{m-2,j}}{2\Delta X}\right). \tag{26a}$$

Thus at the end of the second half time step the resulting tridiagonal system is formed using equations (24a), (25a), and (26a).

Formulation of the vorticity equation in tridiagonal form is simplified since the equation does not contain time directly, thus:

$$\Psi'_{i-1,j}\left(\frac{1}{(\Delta X)^2}\right)-2\,\Psi'_{i,j}\left(\frac{(\Delta y)^2+(\Delta X)^2}{(\Delta X)^2(\Delta y)^2}\right)+\Psi'_{i+1,j}\left(\frac{1}{(\Delta X)^2}\right)=$$

$$-S'_{i,j}-\Psi_{i,j-1}\left(\frac{1}{(\Delta y)^2}\right)-\Psi_{i,j+1}\left(\frac{1}{(\Delta y)^2}\right) \tag{27a}$$

which holds for $\quad 2 \leqslant i \leqslant m-1$
$\qquad\qquad\qquad\quad 2 \leqslant j \leqslant n-1$.

For the case $i = 2$, $2 \leqslant j \leqslant n-1$ noting that $\Psi'_{1,j} = 0$:

$$-2\,\Psi'_{2,j}\left(\frac{(\Delta y)^2+(\Delta X)^2}{(\Delta X)^2(\Delta y)^2}\right)+\Psi'_{3,j}\left(\frac{1}{(\Delta X)^2}\right)=$$

$$-S'_{2,j}-\Psi_{2,j-1}\left(\frac{1}{(\Delta y)^2}\right)-\Psi_{2,j+1}\left(\frac{1}{(\Delta y)^2}\right). \tag{28a}$$

Similarly for $i = m - 1$, $2 \leqslant j \leqslant n - 1$ noting that $\Psi'_{m,j} = 0$

$$\Psi'_{m-2,j}\left(\frac{1}{(\Delta X)^2}\right)-2\,\Psi'_{m-1,j}\left(\frac{(\Delta y)^2+(\Delta X)^2}{(\Delta X)^2(\Delta y)^2}\right)=$$

$$S'_{m-1,j}-\Psi_{m-1,j-1}\left(\frac{1}{(\Delta y)^2}\right)-\Psi_{m-1,j+1}\left(\frac{1}{(\Delta y)^2}\right). \tag{29a}$$

Equations (27a), (28a), and (29a) form the tridiagonal system from which new values of the stream function may be determined.

In determining values of the velocity components u and v it is necessary to consider the first order partial derivatives

$$\frac{\partial\Psi}{\partial y} \text{ and } \frac{\partial\Psi}{\partial x}$$

By considering a Taylor's series expansion for $\Psi'_{i+2,j}$, $\Psi'_{i+1,j}$, $\Psi'_{i-2,j}$ and $\Psi'_{i-1,j}$ about their value at $\Psi'_{i,j}$ it can be shown that the following approximations are valid:

$$\left(\frac{\partial\Psi'}{\partial x}\right)_{i,j} = \frac{-2\Psi'_{i-1,j} - 3\Psi'_{i,j} + 6\Psi'_{i+1,j} - \Psi'_{i+2,j}}{6\Delta x} \tag{30a}$$

and

$$\left(\frac{\partial\Psi'}{\partial x}\right)_{i,j} = \frac{\Psi'_{i-2,j} - 8\Psi'_{i-1,j} + 8\Psi'_{i+1,j} - \Psi'_{i+2,j}}{12\Delta x} \tag{31a}$$

Similar expressions may be developed for $\frac{\partial\Psi}{\partial y}$. Equation (31a) gives fourth order truncation error while equation (30a) gives third order truncation error. Whenever possible equation (31a) will be used to obtain values for velocity components. Considering the v components of velocity where

$$v'_{i,j} = -\left[\frac{\partial\Psi'}{\partial x}\right]_{i,j} \quad :$$

for $i = 2$, $2 \leq j \leq n - 1$ noting that $\Psi'_{1,j} = 0$

$$v'_{2,j} = \frac{3\Psi'_{2,j} - 6\Psi'_{3,j} + \Psi'_{4,j}}{6\Delta x} \tag{32a}$$

for $i = m - 1$, $2 \leq j \leq n - 1$ noting that $v'_{2,j} = -v'_{m-1,j}$ from symmetry conditions and also that $\Psi'_{m,j} = 0$

$$v'_{m-1,j} = \frac{-3\,\Psi'_{m-1,j} + 6\,\Psi'_{m-2,j} - \Psi'_{m-3,j}}{6\,\Delta x} \tag{33a}$$

and finally for $3 \leqslant i \leqslant m - 2$, $2 \leqslant j \leqslant n - 1$

$$v'_{i,j} = \frac{-8\,\Psi'_{i+1,j} + 8\,\Psi'_{i-1,j} - \Psi'_{i-2,j} + \Psi'_{i+2,j}}{12\,\Delta x} \tag{34a}$$

Similar expressions may be developed for $u'_{i,j} = \left[\frac{\partial \Psi'}{\partial y}\right]_{i,j}$

giving for $2 \leqslant i \leqslant m - 1$, $j = 2$

$$u'_{i,2} = \frac{-3\,\Psi'_{i,2} + 6\,\Psi'_{i,3} - \Psi'_{i,4}}{6\,\Delta y} \tag{35a}$$

or $2 \leqslant i \leqslant m - 1$, $j = n - 1$

$$u'_{i,m-1} = \frac{3\,\Psi'_{i,m-1} - 6\,\Psi'_{i,m-2} + \Psi'_{i,m-3}}{6\,\Delta y} \tag{36a}$$

and for $2 \leqslant i \leqslant m - 1$, $3 \leqslant j \leqslant n - 2$

$$u'_{i,j} = \frac{8\,\Psi'_{i,j+1} - 8\,\Psi'_{i,j-1} + \Psi'_{i,j-2} - \Psi'_{i,j+2}}{12\,\Delta y} \quad . \tag{37a}$$

During the next step, new boundary values of the vorticity will be obtained. By considering a Taylor's series expansion for the stream function at points $\Psi_{i,2}$ and $\Psi_{i,3}$ about its value at $\Psi_{i,1}$, on the wall, it can be shown that

$$S'_{i,1} = \frac{-8\,\Psi'_{i,2} + \Psi'_{i,3}}{2\,(\Delta y)^2} \tag{38a}$$

where the fact has been used that $\Psi'_{i,1} = 0$, $\left[\frac{\partial \Psi'}{\partial y}\right]_{i,1} = u_{i,1} = 0$ and

$$\left(\frac{\partial^2 \Psi'}{\partial x^2}\right)_{i,1} = \frac{2\,(\Psi'_{i+1,1} - \Psi'_{i,1})}{(\Delta x)^2} = 0 \qquad \text{along}$$

the wall. The same method is employed to obtain expressions for

$\zeta'_{i,m}$, $\zeta'_{1,j}$ and $\zeta'_{m,j}$ thus for $1 \leqslant i \leqslant m$, $j = n$

$$\zeta'_{i,m} = \frac{-8\,\Psi'_{i,m-1} + \Psi'_{i,m-2}}{2(\Delta y)^2} \tag{39a}$$

for $i = 1$, $1 \leqslant j \leqslant n$

$$\zeta'_{1,j} = \frac{-8\,\Psi'_{2,j} + \Psi'_{3,j}}{2(\Delta x)^2} \tag{40a}$$

and for $i = m$, $1 \leqslant j \leqslant n$

$$\zeta'_{m,j} = \frac{-8\,\Psi'_{m-1,j} + \Psi'_{m-2,j}}{2(\Delta x)^2} \tag{41a}$$

In order to compute a mean value for the Nusselt number it is necessary to obtain an expression for the temperature gradient at the wall

$$\left[\frac{\partial T}{\partial x}\right]_{1,j} .$$

By considering appropriate Taylor's series expansions it can be shown that

$$\left(\frac{\partial T'}{\partial x}\right)_{1,j} = \frac{-11\,T'_{1,j} + 18\,T'_{2,j} - 9\,T'_{3,j} + 2\,T'_{4,j}}{6\,\Delta x} \tag{42a}$$

The mean value for the Nusselt number will be obtained by computing all values of $\left[\frac{\partial T}{\partial x}\right]_{1,j}$ for $1 \leqslant j \leqslant n$ and then using Simpson's rule to obtain a mean value for $\left[\frac{\partial T}{\partial x}\right]_{1,j}$ or $\overline{\left[\frac{\partial T}{\partial x}\right]}_{1,j}$. The average value for the Nusselt number is then given by

$$\overline{Nu} = -\overline{\left(\frac{\partial T}{\partial x}\right)}_{1,j} . \tag{43a}$$

The computational procedures to be utilized in solving the above system of partial differential equations governing convective heat transfer in a rectangular enclosure subjected to transverse vibration will now be given in the sequence in which the computations are to be performed:

(1) Solve the tridiagonal systems composed of equations (8a) through (19a) to determine new values of temperature at the advanced point in time.

(2) Solve the tridiagonal systems composed of equations (21a) through (26a) to determine new values of interior vorticities at the advanced point in time.

(3) Using equations (27a) through (29a) compute new values for the stream function at the advanced point in time.

(4) Using equations (32a) through (37a) compute new values for the velocity components u and v.

(5) Use equations (38a) through (41a) to compute new values of the vorticity on the boundaries.

(6) Using equations (42a) and (43a) compute the mean Nusselt number.

The sequence (1) through (6) is applied repeatedly until a steady state condition is reached.

Although associated problem formulation and programming using the implicit alternating direction method is more complicated than for either an implicit or explicit method alone, there are two main advantages of this method. These advantages are:

(1) The m by n tridiagonal matrix system is more easily

solved than is the general m by n system arising in the straight explicit or implicit method.

(2) Larger time increments may be utilized in the implicit alternating direction method as compared to the explicit or implicit method, thus allowing a considerable saving in computer time.

NATURAL CONVECTION IN ENCLOSURES AND THE EFFECTS OF VIBRATION ON NATURAL CONVECTION

1. ABBOT, M.R., "A Numerical Method for Solving the Equations of Natural Convection in a Narrow Concentric Cylindrical Annulus with a Horizontal Axis," Quarterly Journal of Mechanics and Applied Mathematics, 17, pp. 471-481 (1964).

2. ANANTANARAYANANR. and A. RAMACHANDRAN, "Effect of Vibration on Heat Transfer from a Wire to Air in Parallel Flow," Trans. ASME, 80, p. 1426 (1958).

3. ANDRES, J. M. and U. INGARD, "Acoustical Streaming at Low Reynolds Numbers," Journal of the Acoustic Society of America, 25, no. 5, pp. 932-938 (Sept. 1953).

4. ANDRES, J. M. and U. INGARD, "Acoustical Streaming at High Reynolds Numbers," Journal of the Acoustic Society of America, 25, no. 5, pp. 923-932 (Sept. 1953).

5. ARPACI, V. S., J. A. CLARK and S. ESHGHY, "The Effect of Longitudinal Oscillations on Free Convection from Vertical Surfaces," ASME Trans., Series J. of Appl. Mechanics, 32, pp. 183-191 (March, 1965).

6. ASATURYAN, A. S. and B. A. TONKOSHKUROV, "Free Thermal Convection Near the Linear Source of Heat," AD 610 372.

7. BARAKAT, H. Z., "Transient Laminar Free-Convection Heat and Mass Transfer in Two-Dimensional Closed Containers Containing Distributed Heat Source," ASME Paper 65-WA/HT-28.

8. BATCHELOR, G. K., "Heat Convection and Buoyancy Effects in Fluids," Quarterly Journal Roy. Meteor Soc., 80, no. 345, pp. 339-348, 1 plate (July 1954).

9. BATCHELOR, G. K., "Heat Transfer by Free Convection Across a Closed Cavity Between Vertical Boundaries at Different Temperatures," Quart. of Applied Mathematics, 12, pp. 209-233 (1954).

10. BECK, J. V., "Calculation of Thermal Diffusivity from Temperature Measurements," J. Heat Transfer, 85, no. 2, p. 181 (1963).

11. BECKMAN, W., "Die Warmeubertrangung in Zylindrischen Gasschichten Bei Naturlicher Konvektion," Forsch Geb. Ingenieurw, 2, pp. 165-178, 212-227, 407 (1931).

12. BERGLES, A. E. and P. H. NEWELL, JR., "The Influence of Ultrasonic Vibrations on Heat Transfer to Water Flowing in Annuli," Int. J. Heat and Mass Transfer, 8, no. 10, pp. 1273-1280 (1965).

13. BINDER, R. C., "The Damping of Large Amplitude Vibrations of a Fluid in a Pipe," Jour. of the Acoustical Soc. of America, 15, pp. 41-43 (1943).

14. BIRD, R. B., "Berekeningsmethoden Voor Het Warmtegeleidingsvermogen Van Gassen En Vloeistoffen," Ingenieur, 70, no. 35, pp. 57-62 (August 29, 1958).

15. BISHOP, E. L., L. R. MACK and J. A. SCANLAN, "Heat Transfer by Natural Convection Between Concentric Spheres," Submitted to International J. of Heat and Mass Transfer (1965).

16. BISHOP, E. H., R. S. KOLFLAT, L. R. MACK and J. A. SCANLAN, "Convective Heat Transfer Between Concentric Spheres," Proc. 1964 Heat Transfer Fluid Mech. Inst., pp. 69-80, Standford Univ. Press (1964).

17. BISHOP, E. H., R. S. KOLFLAT, L. R. MACK and J. A. SCANLAN, "Photographic Studies of Convection Patterns between Concentric Spheres," Soc. Photo-Optical Instr. Engrs. J., 3, pp. 47-49 (1964-1965).

18. BJORKLUND, I. S. and W. M. KAYS, Trans. ASME J. Heat Transfer, 81, Series C, 175 (1959). Concerns heat convection in air enclosed by two rotating cylinders.

19. BLANKNESHIP, V. D., "The Influence of Transverse Harmonic Oscillations on the Heat Transfer from Finite and Infinite Vertical Plates in Free Convection," PhD Thesis, Univ. of Michigan, Ann Arbor (August 1962).

20. BLANKENSHIP, V. D. and J. A. CLARK, "Experimental Effects of Transverse Oscillations on Free Convection of a Vertical, Finite Plate," Trans. Amer. Soc. Mech. Engrs. Series C, Journal of Heat Transfer, 86, 2, pp. 159-165 (May 1964).

21. BLANKENSHIP, V. D. and J. A. CLARK, "Effects of Oscillation on Free Convection from a Vertical Finite Plate," Trans. Amer. Soc. Mech. Engrs. Series C, J. of Heat Transfer, 86, no. 2, pp. 149-158 (May 1964).

22. BOELTER, L. M. K. and W. E. MASON, "Vibration--Its Effect on Heat Transfer, Power Plant Engr., 44, pp. 43-46 (1940).

23. BOGGS, J. H. and W. L. SIBBITT, "Thermal Conductivity Meas. of Viscous Liquids," Indust. Engr. Chem., 47, 2, 289-293 (Feb. 1955).

24. BOLOTINA, K. S., "On Usloviyakh Vozniknoveniya Svobodnoi Knoveksii V Kanale Pryamougol 'nogo Secheniyq," Akademiya Wauk USSR, Izvestiya, Otdelenie Teknicheskikh Nauk, Mekhanika i Mashinostreonie, no. 1, pp. 73-6 (Jan.-Feb. 1962). (Conditions involving free convection in rectangular channel, asympotic solution presented for passage from molecular heat transfer to free convection in vertical rectangular channel.)

25. BORISOV, YU. YA. and YU. G. STATNIKOV, "Flow Currents Generated in an Acoustic Standing Wave," Jour. of Acoustics (USSR) Vol. 11, no. 1, (Jan.-Mar. 1965).

26. BROOKS, R. V., "Free Convection Velocity Measurements by the Use of Neutral Density Particles," Thesis, Georgia Institute of Technology (1965).

27. BUGAENKO, G. A., "Free Thermal Convection in Vertical Cylinders of Arbitrary Cross Sections," (in Russian) Pril. Mat. Mekh., 17, no. 4, pp. 496-500 (July-August 1953).

28. BURDOKOV, A. P. and V. E. NAKORIZKOV, "Heat Transfer from a Cylinder in a Sound Field at Grashof Numbers Tending to Zero," Teploobmen at Tsilindra v Zvukovom Pole pri Priklandoi nekhaniki I Teknicheskoi Fiziki, pp. 119-124 (Jan.-Feb. 1965). (in Russian)

29. BYKOV, L. T., "Estimation of the Rate of Displacement of Air in a Confined Space with Natural Convection," Jour. of Engineering Physics (Russian) 8, no. 2 (Feb. 1965).

30. BYKOV, L. T. and V. V. MALOZEMOV, "Some Temperature Distribution Relations for Confined Spaces with Internal Heat Sources," Jour. of Engr. Physics (Russian), 8, no. 2 (Feb. 1965).

31. CARLSON, W. O., "Interferometric Studies of Convective Flow Phenomena in Vertical Plane Enclosed Air Layers," PhD Thesis, Graduate School, Univ. of Minn. (1956).

32. CAVALLARO, L., A. INCELLI and G. PANCALDI, "On Some Improvements on a Cryoscopic Precision Apparatus and on the Control of a Thermocouple," (in Italian) Ric. Sci., 23, no. 12, pp. 2237-2243 (Dec. 1953).

33. CHANDRASEKHAR, S., Proc. Roy. Soc. (London) A237, p. 476 (1956). Free convection in thin layers heated from below.

34. CHERVIAKOV, S. S., "Experimental Investigation of the Influence of Vibration of a Sphere on Heat and Mass Exchange in a Turbulent Stream of Air," In Its J. of Engr. Phys. Misk, 6, no. 6, pp. 27034 (June 1963, 17 Dec. 1963). See N64-11965 03=01 OTS $3.50.

35. CHERVIAKOV, S. S., "Experimental Investigation of the Effect of Vibrations on the Heat and Mass Transfer of a Cylinder and a Cone in Turbulent Air Flow," Inzhenerno-Fizicheskii Zhurnal, 6, pp. 10-21 (Aug. 1963). (in Russian)

36. COULBERT, C. D., "Mach-Zehnder Interferometer Applications as Used in the Study of Convection and Conduction Heat-Transfer Systems," ASME Annual Meeting, New York, (Dec. 1952) Paper No. 52-A-9, 3 pp. 13 Figs.

37. CRANDALL, I. B., "Theory of Vibrating Systems and Sound," D. Van Nostrand, Co., Inc., New York, pp. 95-103 and 229-241 (1927).

38. CRAWFORD, L. and R. LEMLICH, "Natural Convection in Horizontal Concentric Cylindrical Annuli," Industrial and Engr. Chemistry Fundamentals, 1, pp. 260-264 (1962).

39. DE GRAFF, J. G. A. and E. F. M. VAN DER HELD, "The Relation Between the Heat Transfer and the Convection Phenomena in Enclosed Plane Layers," Appl. Sci. Res. (A), 3, no. 6, pp. 393-409 (1953).

40. DEAVER, F. K., W. R. PENNEY and T. B. JEFFERSON, "Heat Transfer from an Oscillating Horizontal Wire to Water," Trans. Amer. Soc. Mech. Engr. Series C, Journal of Heat Transfer, 84, pp. 251-256 (1962).

41. DECKER, A. S., "Natural Convection Equipment," ASHRAE, J 7, no. 4, p. 45 (1965).

42. DONALDSON, I. G., "Free Convection in a Vertical Tube with a Linear Wall Temperature Gradient," Austral. J. Phys., 14, p. 529 (1961).

43. DOUGALL, R. S., T. CHIANG and R. M. FAND, "A study of the Differential Equations of Coupled Vibrations and Free Convection from a Heated Horizontal Cylinder," Wright-Patterson AFB, Ohio, Aeronautical Research Lab, p. 24, 46 (Dec. 1961). Refs. ARL-148 Part I.

44. DRAKHLIN, E., "On Heat Convection in a Spherical Cavity," (in Russian), Zh. Tekh. Fiz., 22, no. 5, pp. 829-831 (May 1952).

45. DROPKIN, D. and E. SOMERSCALES, "Heat Transfer by Natural Convection in Liquids Confined by Two Parallel Plates which Are Inclined at Various Angles with Respect to the Horizontal," Journal of Heat Transfer C 87, pp. 77-84 (1965).

46. DUSINBERRE, G. M., "A Note on the Implicit Method for Finite-Difference Heat Transfer Calculations," ASME Trans., J. Heat Transfer, Series C, pp. 94-95 (Feb. 1961).

47. ECKART, C., "Vortices and Streams Caused by Sound Waves," The Physical Review, 73, no. 1, pp. 68-76 (Jan. 1, 1948).

48. ECKERT, E. R. G. and W. O. CARLSON, "Natural Convection in Air Layer Enclosed Between Two Vertical Plates with Different Temperatures," Inter. J. of Heat and Mass Transfer, 2, pp. 106-120 (1961).

49. EICHHORN, R., "Flow Visualization and Velocity Meas. in Natural Conv. with Tellurium Die Method," ASME Trans., J. Heat Transfer, 83, Series C, pp. 379-381.

50. EICHHORN, R., "Measurement of Low Speed Gas Flows by Particle Trajectories: A New Determination of Free Convection Velocity Profiles," Int. J. Heat Mass Transfer, 5, p. 915 (1962).

51. ELDER, J. W., "Laminar Free Convection in a Vertical Slot," J. Fluid Mech. 23, Part I, pp. 77-98 (1965).

52. ELDER, J. W., "Turbulent Free Convection in a Vertical Slot," J. Fluid Mech. 23, Part I, pp. 99-111 (1965).

53. EMERY, A. and N. C. CHU, "Heat Transfer Across Vertical Layers," J. of Heat Transfer, C 87, pp. 110-116 (1965).

54. ESHGHY, S., V. S. ARPACI and J. A. CLARK, "The Effect of Longitudinal Oscillations on Fluid Flow and Heat Transfer from Vertical Surfaces in Free Convection". Paper in preparation, also Tech. Report No. 1, DRA Project 05065, Univ. of Michigan, Ann Arbor (June 1963).

55. FAND, R. M. and J. KAYE, "Influence of Sound on Free Convection from Horizontal Cylinder," Trans. ASME, J. Heat Transfer, 83, Series C, no. 2, pp. 133-148 (May 1961).

56. FAND, R. M. and J. KAYE, "Effects of High Intensity Stationary & Progressive Sound Fields on Free Convection from a Horizontal Cylinder," TN 59-18 (1959).

57. FAND, R. M. and J. KAYE, "The Influence of Vertical Vibration on Heat Transfer in Free Convection from a Horizontal Cylinder," Intern. Dev. in Heat Transfer, ASME, pp. 490-498 (1961).

58. FAND, R. M. and J. KAYE, "Acoustic Streaming Near a Heated Cylinder," Journal of Acoustical Soc. of America, 32, p. 579 (1960).

59. FAND, R. M. and J. KAYE, "The Influence of Sound on Free Convection from a Horizontal Cylinder," ASME Paper No. 60-HT-14.

60. FAND, R. M., "On the Mechanism of Interaction Between Vibrations and Heat Transfer," Wright Patterson AFB, Ohio Aeronautical Research Lab., ARL-148, Part IV, p. 36 (December 1961).

61. FAND, R. M. and OTHERS, "The Influence of Sound on Heat Transfer from a Cylinder in Crossflow," Inter. J. of Heat and Mass Transfer, 5, pp. 571-596 (July 1963).

62. FAND, R. M., "The Influence of Acoustical Vibrations on Convective Heat Transfer to Liquids," Office of Saline Water, Res. and Dev. Report No. 89, US Dept of Interior, PB181584 (1964).

63. FENSTER, S. K., G. J. VAN WYLEN and J. A. CLARK, "Transient Phenomena Associated with the Pressurization of Liquid Nitrogen Boiling at Constant Heat Flux," Advances in Cryogenic Engr. 5, pp. 226-234.

64. FORSYTHE, G. E and W. R. WASOW, Finite Difference Methods for Partial Differential Equations, John Wiley and Sons, Inc., New York (1960).

65. FULTZ, D. and Y. NAKAGAWA, Proc. Roy. Soc. (London), A231, p. 211 (1955).

66. GERSHUNI, G. Z., "On Free Convection in Space Between Vertical Coaxial Cylinders," Dokladi Akad, Nauk USSR, N. S., 86, 4, pp. 698-699 (Oct. 1952). (in Russian)

67. GOODY, R. M., "The Influence of Radiative Transfer on Cellular Convection," J. Fluid Mechanics, 1, p. 424 (1956).

68. HAMMITT, F. G., "Heat and Mass Transfer in Closed, Vertical, Cylindrical Vessels with Internal Heat Sources for Homogeneous Reactors," PhD Thesis, Univ. of Mich. (Dec. 1957).

69. HAN, L. S., "Laminar Heat Transfer in Rectangular Channels," ASME Trans. J. Heat Transfer, 81, Series C, no. 2, pp. 121-128 (May 1959).

70. HARDEN, D. G. and J. H. BOGGS, "Transient Flow Characteristics of Nat. Circulation Loop Operated in Critical Region," Heat Transfer and Fluid Mechanics Institute-Proc. Preprints for Meeting June 10-12, pp. 38-50 (1964).

71. HARJE, D. T., "Effect of Oscillating Flow on Heat Transfer in a Tube," Progress Report No. 20-362, JPL, Cal. Inst. of Tech. (August 1958).

72. HARTNETT, J. P., W. E. WELCH and F. W. LARSEN, "Free Convection Heat Transfer to Water and Mercury in an Enclosed Cylindrical Tube," Nuclear Engr. and Science Conference, Preprint 27, Session XX, Chicago (March 17-21, 1958).

73. HARTNETT, J. P. and W. E. WELCH, "Experimental Studies of Free Convection Heat Transfer in a Vertical Tube with Uniform Heat Flux," ASME Trans., 79, p. 1961 (1957).

74. HEATON, H. S., W. C. REYNOLDS and W. M. KAYE, "Heat Transfer in Annular Passages. "Simultaneous development of velocity and temperature fields in laminar flow," International J. Heat Mass Transfer, 7, no. 7, pp. 763-781 (July 1964).

75. HELLUMS, J. S., "Finite-Difference Computation of Natural Convection Flow," PhD Thesis, Univ. of Mich. (Sept. 1960).

76. HENSHAW, D. H. and D. F. DAW, "Design of Cold Temperature Probes," Nat. Aero. Estab. Cana. LR-184 (January 1, 1954).

77. HOLMAN, J. P., H. E. GARTRELL and F. E. SOEHNGEN, "A Study of Free Convection Boundary Layer Oscillations and their Effects on Heat Transfer," WADC Tech. Report 59-3 (December 1959)

78. HOLMAN, J. P. and T. P. MOTT SMITH, "The Effects of High Constant Pressure Sound Fields on Free Convection Heat Transfer from a Horizontal Cylinder," WADC Tech. Note 58-352 ASTIA Doc. no. 206906, Aero. Res. Lab. (December 1958)

79. HOLMAN, J. P., J. Heat Transfer, C 82, p. 393 (1960). Concerns a qualitative explanation of the change in heat transfer coefficients above a certain critical sound pressure.

80. HOLMAN, J. P., "The Mechanism of Sound Field Effects on Heat Transfer," J. of Heat Transfer, pp. 393-6 (November 1960).

81. ISAKOFF, S. E., "Analysis of Unsteady Flow Using Direct Electrical Analogs," Ind. and Engr. Chem., 47, no. 3, pp. 413-421 (March 1955).

82. IZUMI, R., "Natural Heat Convection Inside the Vertical Tube," Proc. 6th Japan Nat. Congr. Appl. Mech., Univ. of Kyoto, Japan, pp. 393-396 (October 1956).

83. JACKSON, T. W., W. B. HARRISON and W. C. BOTELER, "Free Convection, Forced Convection, and Acoustic Vibrations in a Constant Temperature Vertical Tube," Trans. ASME, J. Heat Transfer, 81, Series C, 68 (1959).

84. JACKSON, T. W., K. R. PURDY and C. C. OLIVER, "The Effects of Resonant Acoustic Vibrations on the Nusselt Numbers for a Constant Temperature Horizontal Tube," Intern. Dev. in Heat Transfer, Boulder, Colorado, Part II, Sec. B, p. 483 (1961).

85. JOHNSON, N. R. and F. OSTERLE, "The Influence of Gradiant Temperature Fields on Thermocouple Measurements," ASME-AIChE Heat Trans. Conf., Univ. Park, Pa., Pap. 57-HT-18, p. 20 (Aug. 1957).

86. JONES, C. D. and D. J. MASSON, "An Interferometric Study of Free-Convection Heat Transfer from Enclosed Isothermal Surfaces," Trans. ASME 77, 8, pp. 1275-1281 (November 1955).

87. KAYS, W. M. and A. L. LONDON, "Convective Heat Transfer and Flow-Friction Behavior of Small Cylindrical Tubes--Circular and Rectangular Cross Sections," Trans. ASME, 74, 7, pp. 1179-1189 (October 1952).

88. KESTIN, J., P. F. MARDER and H. E. WANG, "On Boundary Layers Associated with Oscillating Streams," Appl. Sci. Res. Section A., 10, no. 1 (1961).

89. KHLEBUTIN, G. N. and G. F. SHAILDUROV, "Heat Convection in a Vertical Annular Tube," (in Russian) Inzhen. Fiz. Zh. 8, no. 1, pp. 3-7 (1965).

90. KRASSOLD, H., "Warmeabgabe Von Zylindrischen Flussigkeitschieten bei Naturlicker Konvektion," Forsch. Geb. Ingenieurw., 5, pp. 136-191 (1943).

91. KRAUS, A. D., "Heat Flow Theory," Elec. Mfg., 63, no. 4, pp. 123-42 (April 1959).

92. KREITH, F., L. G. ROBERTS, J. A. SULLIVAN and S. N. SINHA, "Convection Heat Transfer and Flow Phenomena of Rotating Spheres," Intern. J. of Heat and Mass Transfer, 6, no. 10, pp. 881-95 (October 1963).

93. KUBANSKII, P. N., "Intesifikatiya Teploobemna Ultrazvukom," Teploenergetika, no. 11, pp. 79-83 (November 1962). (Effect of high amplitude acoustic oscillations on convective heat transfer.)

94. KUDRIASHEV, L. I. and IU. A. TKACHEV, "Investigation of the Effect of Body Vibrations on the Heat Transfer Coefficients in Conditions of Free Convection," Issledovanic uliianiia vibratsii Tela na Koeffitsient teplootdachi v usloviakh svobodnoi knovektsii. Aviatsionnaia Teknika, 8, no. 1, pp. 54-72 (1965). (in Russian).

95. LARSON, M. B. and A. L. LONDON, "Study of Effects of Ultrasonic Vibrations on Convective Heat Transfer to Liquids," ASME Paper 62-HT-44, p. 16, (August 5-8, 1962).

96. LARSON, M. B., "Study of Effects of Ultrasonic Vibrations on Convective Heat Transfer in Liquids," Stanford University Dept. of Mech. Engr. Tech. Report, 48, p. 102 (September 30, 1960).

97. LARSEN, F. W. and J. P. HARTNETT, "Effect of Aspect Ratio and Tube Orientation on Free Convection Heat Transfer to Water and Mercury in Enclosed Circular Tubes," Trans. ASME, J. Heat Transfer, 83, Series C, no. 1, pp. 87-93 (February 1961).

98. LEE, B. H. and P. S. RICHARDSON, "Effect of Sound on Heat Transfer from a Horizontal Cylinder at Large Wavelengths," J. Mech. Engr. Sci., 7, no. 2, p. 127 (1965).

99. LEMLICH, ROBERT and M. A. RAO, "The Effect of Transverse Vibration on Free Convection from a Horizontal Cylinder," Int. J. Heat Mass Transfer, 8, pp. 27-33 (1965).

100. LEMLICH, R. and H. W. HWU, "Effect of Acoustic Vibration on Forced Convective Heat Transfer," A. I. Chem. Engr. Journal, 7, no. 1, pp. 102-106 (March 1961).

101. LEMLICH, R. and M. R. LEVY, "Effect of Vibration on Natural Convective Mass Transfer," Journal of American Inst. Chem. Engrs., 7, pp. 240-242 (1961). Supporting data filed in Chem. Engr. Dept. of Univ. of Cincinnati, Cincinnati, Ohio.

102. LEMLICH, ROBERT, "Effect of Vibration on Natural Convective Heat Transfer," Industrial & Engr. Chem., 47, no. 6, pp. 1175-1180 (1955).

103. LEMLICH, R., "Vibration and Pulsation Boost Heat Transfer," Chem. Engr., 68, pp. 171-176 (1961).

104. LEVY, S., "Integral Methods in Natural Convection Flow," ASME Paper No. 55, APM 22.

105. LEVY, J. J., Journal of Applied Mech., 22, p. 515 (1955). (Obtains solutions for free convection flows in enclosed tubes using the integral method.)

106. LIGHTHILL, M. J., "Theoretical Consideration Free Convection in Tubes," Quart. J. Mech. and Appl. Math., 6, pp. 398-439 (1953).

107. LIGHTHILL, M. J., "The Response of Laminar Skin Friction and Heat Transfer to Fluctuations in Stream Velocity," Proc. of Royal Soc., Series A, 224 p. 23 (1954).

108. LIN, C. C., "Motion of the Boundary Layer with a Rapidly Oscillating External Flow," Proc. 9th Intern. Congr. of Appl. Mech., 4, (1957).

109. LINKE, W., "Warmeubergang bei pulsierender Stromung," Zeit. des VDI, Bd, 95, p. 1179 (1953).

110. LIU, C. Y., W. K. MUELLER and F. LANDIS, "Natural Convection Heat Transfer in Long Horizontal Cylindrical Annuli," Inter. Dev. in Heat Transfer, Part V, American Soc. Mech. Engrs., New York, pp. 976-984 (1961).

111. LORD RAYLEIGH, "Theory of Sound," 1st American Edition (1st Ed. 1877) II, Dover Pub. New York, App. A (1945).

112. MARKHAM, J. J., "Second-Order Acoustic Fields: Streaming with Viscosity and Relaxation," The Physical Review, 86, no. 4, pp. 497-502 (March 15, 1952).

113. MARTINELLI, R. C., L. M. K. BOELTER, E. B. WEINBERG and S. YAKAHI, "Heat Transfer to a Fluid Flowing Periodically at Low Frequencies in a Vertical Tube," Trans. ASME, 65, pp. 789-798 (1943).

114. MARTINELLI, R. C. and M. K. BOELTER, "The Effect of Vibration on Heat Transfer by Free Convection from a Horizontal Cylinder," Proc. 5th Int. Cong. Appl. Mech., p. 578 (1938).

115. MARTINI, W. R. and S. W. CHURCHILL, "Natural Convection Inside a Horizontal Circular Cylinder," AIChE Journal, 6, p. 251 (1960).

116. MAWARDI, O. K., "On Acoustic Boundary-Layer Heating," Journal of the Acous. Soc. of America, 26, no. 5, pp. 726-731 (Sept. 1954).

117. MEDWIN, H., "An Acoustic Streaming Experiment in Gases," Journal of the Acous. Soc. of America, 26, no. 3, pp. 332-341 (May 1954).

118. MELDAHL, A., "The Heat Transfer Coefficient for Fluids in Laminar Movement," Engr., 169, 4398, p. 541 (May 1950).

119. MERTE, H., JR. and J. A. CLARK, "Pool Boiling in Accelerating System," Trans. ASME, J. Heat Transfer, 83, no. 3, pp. 233-43 (August 1961).

120. MIKULIN, E. I., "Temperaturnoe pole dvokh tverdykh tel rzdelennkh zazorom," Inzhereno fizicheskii zhrnal, 4, no. 2, pp. 52-7 (Feb. 1961). (Analytical solution of temperature distribution in 2 concentrically located cylinders of finite length separated by gap, when heating (cooling) begins at end of cylinders.)

121. MINORSKY, N., "Self-Excited Oscillations in Dynamical Systems Possessing Retarded Actions," Jour. of Applied Math., 9, no. 2, 265-271 (1942).

122. MORGAN, P. G., "Effect of Vibration and Pulsation on Heat Transfer," Engr. & Boiler House Rev., 78, no. 4, pp. 128-129 (April 1963).

123. MORKOVIN, M. V., "On Eddy Diffusivity Qusi-Similarity and Diffusion Experiments in Turbulent Boundary Layers," Int. J. Heat Mass Transfer, 8, no. 1, p. 129 (1965).

124. MORSE, PHILIP M., "Plane Waves of Sound," Vibration & Sound, Second Edition, pp. 217-229 (energy in a plane wave).

125. MURAKAWA, K., "Heat Transfer by Free Convection in Closed Vertical Pipes with Annular Spacing," (in Japanese) Trans. Japan Soc. Mech. Engrs., 20, pp. 100, 797-802 (1954).

126. MYATT, D. O., ET ALL, "Pulsatory and Vibrational Phenomena," Ind. and Engr. Chem., 47, no. 6, pp. 1142-1164 (June 1955).

127. NAKAGAWA, Y., Proc. Roy. Soc. (London), A240, p. 108 (1957). (Free convection studies in thin layers heated from below are reported.)

128. NAKAGAWA, Y., "Heat Transport by Convection," Physics of Fluids, 3, no. 1, pp. 82-86 (Jan.-Feb. 1960).

129. NANDA, R. S. and V. P. SHARMA, "Free Convection Laminar Boundary Layers Oscillating Flow," J. of Fluid Mech., 15, part 3, pp. 419-428 (1963).

130. NEELY, D. F., "Effect of Vibration on Heat Transfer from Cylinders in Free Convection," MS Thesis, 77 GA/ME/64-2 AD-61-173, N65027290, Air Force Inst. of Tech., Wright-Patterson AFB, Ohio (August 1964).

131. NYBORG, W. L., "Acoustic Streaming Due to Attenuated Plane Waves," J. of the Acous. Soc. of America, 25, no. 1, pp. 68-75 (1953).

132. OSTRACH, S. and P. R. THORNTON, Trans. ASME, 80, p. 363 (1958).

133. OSTRACH, S. and P. R. THORNTON, "On the Stagnation of Natural Convection Flows in Closed End Tubes," ASME Paper no. 57-SA-2.

134. OSTRACH, S., Trans. ASME, 79, p. 299 (1957).

135. OSTRACH, S., "An Analysis of Laminar Free Convection Flow and Heat Transfer about a Flat Plate Parallel to the Direction of the Generating Body Force," Naca Report 1111 (1953).

136. OSTROMOV, G. A., "Free Convection Under the Conditions of Internal Vibration Problems," Naca TM-1407.

137. PELLEW, A. and R. V. SOUTHWELL, "On Maintained Convection Motion in a Fluid Heated from Below," Proc. Roy. Soc., A, 176, 312 (1940).

138. PERKINS, H. C., JR. and G. LEPPERT, "Local Heat Transfer Coefficients on Uniformly Heated Cylinder," Intern. Journal of Heat and Mass Transfer, 7, pp. 153-158 (1964).

139. PERSEN, L. N., "A Simplified Approach to the Influence of Gortler-Type Vortices on the Heat-Transfer from a Wall," ARL 65-88, Office of Aerospace Research, U. S. Air Force, Wright-patterson AFB, Ohio (1965).

140. PFEEFER, R., "Heat and Mass Transport in Multiparticle Systems," I & EC Fundamentals, 3, no. 4, p. 380 (1964).

141. PIERRE, C. ST and C. TIEN, "Experimental Investigation of Natural Convection Heat Transfer in Confined Space for Non-Newtonian Fluid," Can. J. Chem. Engr., 41, no. 3, pp. 122-127 (June 1963).

142. POTTS, G., "Heat Transfer by Laminar Free Convection in Enclosed Plane Gas Layers," Quart. J. of Mech. and Appl. Math., 11, pp. 257-273 (1958).

143. PREDVODITELEV, A. S., "O Khanktere Teplovogo Dvizheniya V Zhidkostyakh," Inzhereno-Fizicheskii Ahurnal, 4, nos. 6, 7, 8, pp. 3-19 (June 1961).

144. RANEY, W. P., J. C. CORELLI and P. J. WESTERVELT, "Acoustic Streaming in the Vicinity of a Cylinder," Jour. of the Acous. Soc. of America, 26, no. 6, pp. 1006-1017 (1954).

145. RICHARDS, R. J., W. G. STEWARD and R. B. JACKOBS, "Survey of Literature of Heat Transfer from Solid Surfaces to Cryogenic Fluids," U. S. Bureau Standards NBS Tech. Note no. 122, p. 44 (October 1961).

146. ROMONOV, A. G., "Study of Heat Exchange in a Dead End Channel Under Free Convection Conditions," Izvestiya Akademiya Nauk, USSR, Otdeleni Teknishes Kikh, no. 6, pp. 63-79 (1956).

147. ROBINSON, A. R., "The Symmetric State of a Rotating Fluid Differentially Heated in the Horizontal," J. Fluid Mech., 6, p. 599 (1959).

148. ROUND, G. F., R. NEWTON and P. J. REDBERGER, "Variable Mesh Size in Iteration Methods of Solving Partial Differential Equations and Appl. to Heat Transfer," Chem. Engr. Progress Symposium, 58, no. 37, pp. 29-42 (1962).

149. RUSS, R. N. and R. S. RUSS, "Effect of Vibration on Heat Transfer from Cylinders in Free Convection," MS Thesis, GA/ME/62-4, AD-292335, N64-80527, Air Force Inst. of Tech., Wright-Patterson AFB, Ohio, p. 86 (August 1962)

150. SCANLAN, J. A., "Effects of Normal Surface Vibration on Laminar Forced Convective Heat Transfer," Indus. & Engr. Chem., 50, no. 10, pp. 1565-1568 (October 1958).

151. SCHLICHTING, H., "Berechung ebener Periodischer Grenzschictstromungen," Phys. zs. 33, p. 327 (1932).

152. SCHMIDT, E., Chem. Engr. Tech. 28, p. 175 (1956).

153. SCHMIDT, R. and W. VON BECKMAN, "Das Temperatur-und Gaschwindigketsfeld vor einer warme abegbenden senkrechten Platte bei Naturiicher Konvektion," Technischi Mechanik und Thermodynamik, 1, pp. 341-349, 391-406 (1930).

154. SCHOENHALS, R. J. and J. A. CLARK, "Laminar Free Convection Boundary-Layer Perturbations Due to Transverse Wall Vibrations," J. Heat Transfer, Series C, 84, p. 225 (1962).

155. SCHOENHALS, R. J., "The Response of Laminar Incompressible Fluid Flow and Heat Transfer to Transverse Wall Vibrations," PhD Thesis, Univ. of Michigan, ASME Paper No. 63-WA-123 (January 1961).

156. SCHOENHALS, R. J. and J. A. CLARK, "Laminar Free Convection Boundary-Layer Perturbations Due to Transverse Wall Vibrations," Trans. ASME Engrs., Series C, J. of Heat Transfer, 86, 2, pp. 159-165 (May 1964).

157. SEIGEL, R. and R. H. NORRIS, "Test of Free Convection in a Partially Enclosed Space Between Two Heated Vertical Plates," ASME Paper no. 56-SA-5, Trans. ASME, 79, p. 663 (1957).

158. SHAILDUROV, G. F., "Convective Heat Transfer in Horizontal Cylinder," Int. J. Heat & Mass Transfer V. 2, no. 4, pp. 280-282 (June 1961).

159. SHINE, A. J., "Effect of Transverse Vibrations on Heat Transfer Rate from Heated Vertical Plate in Free Convection," ASME Paper no. 58-HT-27, p. 8 (August 9-12, 1959).

160. SLAVONOVA, E. I., "Ob Yasheistoi Strukture Konvektivnogo Potoka Zhidkosti V Vertikal 'nom Tsilindre Kruglogo Secheniya," Inzhereno-Fizicheskii Zhurnal, 4, no. 8, pp. 80-86 (August 1961).

161. SOROKIN, V. S., "Steady Motion of a Fluid Heated From Below," Prikl. Mat. Melsh., 18, 2, pp. 197-204 (Mar.-April 1954).

162. SPARROW, E. N. and S. J. KAUFMANN, "Visual Study of Free Convection in a Narrow Vertical Enclosure," NACA RM E55 L, 14-a (Feb. 1956).

163. SPROTT, A. L., J. P. HOLMAN and F. L. DURAND, "Experimental Study of Effects of Strong Progressive Sound Fields on Free Convection Heat Transfer from Horizontal Cylinder," ASME Paper 60-HT-19, p. 12 (August 15-17, 1960).

164. SPURLOCK, J. M., T. W. JACKSON, K. R. PURDY, C. V. OLIVER and H. L. JOHNSON, "The Effects of Resonant Acoustic Vibrations on Heat Transfer to Air in Horizontal Tubes," WADE Tech. Note 59-330 (June 1959).

165. ST. PIERRE, CARL and CHI TIEN, "Experimental Investigation of Natural Convection Heat Transfer in Confined Space for Non-Newtonian Fluid," The Canadian J. of Chem. Engr., pp. 122-127 (June 1963).

166. STRAUMANN, W., "Ein Instationaeres Verfahren Zur Messung Der Waermeleitfaehigkeit Von Fluessigkeiten und Gasen," Schweizer Archiv, 27, no. 7, pp. 290-304 (July 1961).

167. SZEBEHELY, V. G., "Generalization of the Dimensionless Frequency Parameter in Unsteady Flows," Report no. 833, David W. Taylor Model Basin, Wash. (November 1952).

168. TACHIBANA, F., S. FUKUI and H. MITSUMRA, "Heat Transfer in Annulus with Inner Rotating Cylinder," Japan Soc. Mech. Engrs., 3, 9, pp. 119-123 (February 1960).

169. TAYLOR, R. D. and J. G. DASH, Phys. Rev., 106, p. 398 (1957).

170. TELEKI, C., R. M. FAND and J. KAYE, "Influence of Vertical Vibrations on the Rate of Heat Transfer from a Horizontal Cylinder in Air," Wright Air Dev. Command, T. N., pp. 59-357 (1960).

171. TIMO, D. P., "Free Convection in Narrow Vertical Sodium Annuli," KAPL-1082, Knolls Atomic Power Lab., General Electric Company, (Contract No. W-31-109 Eng. 52) (March 5, 1954).

172. TRILLING, L., "On Thermally Induced Sound Fields," Jour. of the Acous. Soc. of America, 27, no. 3, pp. 423-431 (May 1955).

173. WATSON, W. J., "Effect of Vibration on Heat Transfer from Cylinders Vibrated Sinusoidally within a Vertical Plane in Free Convection," MS Thesis, Air Force Inst. of Tech., Wright-Patterson AFB, Ohio, GAW/ME/65-3, AD-623617, CFSTI, N66-14210, p. 58 (1964).

174. WEINBAUM, S., "Natural Convection in Horizontal Circular Cylinder," J. of Fluid Mech., 18, 3, pp. 409-437 (March 1964).

175. WEST, F. B. and A. T. TAYLOR, "The Effect of Pulsations on Heat Transfer," Chem. Engr. Progr., 48, no. 39 (1952).

176. WESTERVELT, P. J., "Effect of Sound Waves on Heat Transfer," Acoustical Soc. America J., 32, 3, pp. 337-338 (March 1960).

177. WESTERVELT, P. J., "Hydrodynamic Flow and Oseen's Approximations," J. Acoustical Soc. of America, 25, p. 951 (1953).

178. WHITT, F. R., "Film Heat-Transfer Coefficients in Vessel Jackets," Brit. Chem. Engr., 6, 8, pp. 533-537 (August 1961).

179. WICK, R. S., "An Analysis of Oscillatory Flow in Liquid-Propellant Rocket Systems and the Application of the Analysis to Combustion Stability," Report no. 20-231, JPL, Pasadena (August 10, 1954).

180. WILKES, J. O., "The Finite Difference Computation of Natural Convection in an Enclosed Rectangular Cavity", PhD Thesis, Univ. Michigan (1963).

181. WILKES, J. O. and S. W. CHURCHILL, "The Finite Difference Computation of Natural Convection in a Rectangular Enclosure," Preprint 10d, 57th Annual Meeting Amer. Inst. Chem. Engrs., Boston (1964).

182. WILLIAMS, J. E. F., "The Noise of High-Speed Missiles,", Random Vibration, 2, pp. 147-176.

183. YAMAGA, J., "Effects of Sounds and Vibrations on Convective Heat Transfer," J. Japan Soc. Mech. Engrs., 65, no. 525, p. 1415 (1962).

184. YANG, W. J. and H. C. YEH, "Unsteady Boundary-Layers on Vibrating Spheres in a Uniform Stream," Physics of Fluids, 8, pp. 806-811 (May 1965).

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185. ABE, A. and P. J. FLORY, "The Thermodynamic Properties of Mixtures of Small, Nonpolar Molecules," J. Amer. Chem. Soc., 87, no. 9, p. 1838 (1965).

186. BEN-NAIM, A., "On the Difference Between the Thermodynamic Behavior of Argon in D_2O and H_2O," J. Chem. Phys., 42, no. 5, p. 1512 (1965).

187. BONDI, A., "On the Thermal Conductivity of Liquids and Polymers and its Relation to the Pressure Coefficient of Viscosity," J. Chem. Phys., 19, no. 1, pp. 128-129 (January 1951).

188. BORN, M. and H. S. GREEN, "A General Kinetic Theory of Liquids," Cambridge Press, Cambridge, England (1949).

189. BRYNGDAHL, O., "Accurate Determination of Coefficients of Thermal Conductivity in Liquids with the Aid of a Shear-Interferometric Method," Ark. Phys., 21, no. 22, p. 289 (1962). (in German)

190. CHALLONER, A. R. and OTHERS, Proc. Roy. Soc. (London), pp. 245, 259 (1958).

191. CHANDRA, NAD V. S. KANDA, "Thermodynamic Properties of He^3 and He^4 Solutions," Phys. Fluids, 7, no. 1, 7 (1964).

192. CODEGONE, C., "Thermal Conductivity Measurements to 180C," Italian, Thrmotecnica, 4, pp. 573-574 (December 1950).

193. COLLINS, D. J., R. GREIF and A. E. BRYSON, JR., "Measurements of the Thermal Conductivity of Helium in the Temperature Range 1600-6700°K," Int. J. Heat Mass Transfer, 8, no. 9, p. 1290 (1965).

194. CONNOLY, J. H., "Combined Effect of Shear Viscosity Thermal Conduction and Thermal Relaxation on Acoustic Propagation in Linear-Molecule Ideal Gases," J. Acoust. Soc. Amer., 36, no. 12, p. 2374 (1964).

195. CUTLER, M. and G. T. CHENEY, "Heat Wave Methods for the Measurement of Thermal Diffusivity," J. Appl. Phys., 34, no. 7, p. 1902 (1963).

196. DAUPHINEE, T. M., D. K. C. MACDONALD and W. B. PEARSON, "The Use of Thermocouples for Measuring Temperatures below 70°K. A New Type of Low Temperature Thermocouple," J. Sci. Instr., 80, nos. 1, 11, pp. 399-400 (November 1953).

197. EISEEL, K. M. and A. C. H. HALLETT, "The Viscosity of Liquid Helium at Frequencies of 11.3 kc/sec and 35.5 kc/sec," Proc. of 5th Intern. on Low Temperature Physics & Chem., Univ. of Wisconsin Press (1958).

198. EISENSTEIN, A. and N. S. GINGRICH, Phys. Rev., 62, p. 261 (1942).

199. ENGLEHARDT, A. G. and A. V. PHELPS, "Transport Coefficients and Cross Sections in Argon and Hydrogen-Argon Mixtures," Phys. Rev., 133, no. 1A, A 375 (1964).

200. EPSTEIN, M., "A Model of the Boltzmann Collision Integral for Mixtures of Light and Heavy Particles," Rept. No. TDR-469 (5240-20)-9, Aerodynamics and Propulsion Research Lab., Laboratory Operations, Aerospace Corp. El Segundo, California (1965).

201. GAYLORD, E. W. and OTHERS, "On the Theoretical Analysis of a Dynamic Thermocopule," Trans. ASME, 80, p. 307 (1958).

202. HALPERN, C. and R. J. MOFFAT, Nat. Bureau of Standards Monogr., 27 (1961).

203. HARPER, J. C., "Coaxial Cylinder Viscometer for Non-Newtonian Fluids," Rev. Sci. Instr., 32, p. 425 (1961).

204. HARRISON, E. F., "Intermolecular-Force Effects on the Thermodynamic Properties of Helium with Application," AIAA J., 2, no. 10, p. 1854 (1964).

205. HECTH, C. E., "Classical Equation of State for Molecular Interacting with Purely Repulsive Potentials," J. Chem. Phys., 42, no. 8, p. 2862 (1965).

206. HEIKKILA, W. J. and A. C. H. HALLETT, "The Viscosity of Liquid Helium II," Can. Jour. Phys., 33, p. 420 (1955).

207. HENSHAW, D. G., Phys. Rev., 105, p. 976 (1957).

208. HERZBERG, G., Spectra of Diatomic Molecules, D. Van Nostrand Company, Inc., Princeton, N. J. (1959).

209. HERZBERT, F., "Effective Density of Boiling Liquid Oxygen," Advances in Cryogenic Engr., 5.

210. HIRSCHFELDER, J., C. F. CURTISS and R. B. BIRD, "Molecular Theory of Gases and Liquids," John Wiley, New York.

211. HOLSEN, J. N. and M. R. STRUNK, "Binary Diffusion Coefficients in Nonpolar Gases," I & EC, Fundamentals, 3, no. 2, p. 143 (1964).